VECTORS

A vector can be used to describe the movement in a translation.

A vector can be defined by two quantities: its size, and its direction. A vector can be placed anywhere in a plane, and is usually represented by a straight line. The **size** of a vector is the length of the line; the **direction** of the vector is how the line is pointing, i.e. in the direction of the arrow.

The **negative** of the vector is a vector that is equal in size but opposite in direction.

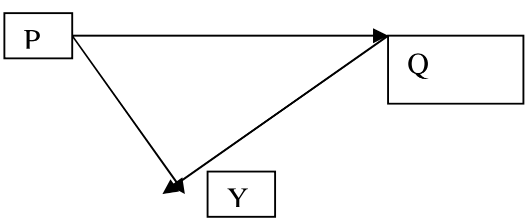
Scalar multiple of a given vector is found by multiplying its size by a single number, leaving its direction unchanged.

You can add vectors by combining them end to end, so that their direction follow on from each other. Adding vectors is commutative because however you combine them, the **resultant** is the same. Adding vectors is associative because however pairs of vectors are grouped, the resultant is the same.

Adding a negative vector can be written as a subtraction. In general, vectors **a-b,** and **b-a** are the same in size, but opposite in direction. Therefore **b-a** is the negative vector of **a-b.**

We refer to a vector by labelling each end. **KL** is a vector with its size equal to the distance from K to L and direction from K to L. **LK** is the same size as **KL** and opposite in direction.

This notation is useful when giving a vector as a vector sum, e.g. **PY = PQ + QY**



We usually use vectors to prove facts in plane geometry – this is a study of the properties and relationships of points and lines in two-dimensional space. This study includes proving that lines are parallel, proving that points are collinear.

Points are **collinear** when they all lie in the same straight line.

Quantities involving both magnitude = size and direction are called **vector** quantities.

Quantities involving only magnitude = size are called **scalar** quantities.

For 2 vectors to be **equal** they must have the same magnitude and the same direction. Thus if we represent vector **a** by a line segment of a certain magnitude and direction, then any other line segment of the same magnitude and direction will also equal **a.**

Two vectors **a** and **b** are **parallel** if one is a scalar multiple of the other, i.e. if $a = \lambda b$.

If λ is positive, the vectors are parallel and in the same direction, i.e. they are *like* parallel vectors.

If λ is negative, the vectors are parallel and in opposite direction, i.e. they are *unlike* parallel vectors.

For two **non-parallel** vectors **a** and **b**, if $\lambda\mathbf{a} + \mu\mathbf{b} = \alpha\mathbf{a} + \beta\mathbf{b}$, then $\lambda = \alpha$, and $\mu = \beta$, i.e. we may equate the coefficients of vector **a** appearing on one side of the equation with those of vector **a** appearing on the other side, and similarly vector **b**.

BOUND VECOTR is a set of all oriented abscissas which have a common original point.

The magnitude = size of oriented abscissa equals to the distance of its marginal points.

FREE VECTOR is a set of oriented non-zero vectors, which have the same direction and the same size, and which represent the same mathematical object called free vector.

Zero oriented abscissas form a zero vector.

Any oriented abscissa, which belongs to a free vector, is called a **representant** of a free vector.

POSITION VECTOR

The position of any point in a plane can be stated in terms of an ordered pair (x,y), i.e. the coordinates of the point. This ordered pair gives the perpendicular distance from the point to each of the coordinate axes with respect to the origin O.

We could also define the point P by giving the distance and direction of P from the origin, i.e. by stating the vector **OP**. This vector is called the **position vector** of the point P. This position vector can also be expressed in terms of its horizontal and vertical components, in one of the following ways:

- as a column matrix, **OP** = $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
- by using unit vectors. A unit vector is a vector of length one unit in a given direction. If we denote a horizontal unit vector by **i** and a vertical unit vector by **j** then

OP = 3i + 2j. If a third dimension is required, we use **k** to represent a unit vector at right angles to the plane containing **i** and **j.**

Any vector lying in the plane of the unit vectors **i** and **j** can be expressed in the form **xi + yj.** The unit vectors **i** and **j** are **base vectors** from which other **coplanar** vectors can be built up. Although it is usually convenient to use **i** and **j** as the base vectors, any pair of non-parallel coplanar vectors **a** and **b** could be used instead.
Thus for three coplanar vectors **a, b,** and **c** it is possible to express any one of the vectors in terms of the other two.

COORDINATES

Let's have a vector **v = AA'**

v = **A' – A**

u = **AB = B – A**

A $[a_1, a_2, a_3]$

B $[b_1, b_2, b_3]$

u $[u_1, u_2, u_3] = [b_1 - a_1 , b_2 - a_2, b_3 - a_3]$

The oriented abscissa **AB** is a position of a vector **u.** The coordinates of vector **u** are real numbers u_1, u_2, u_3.

Multiple of a vector

u=AB, k **u** = k **AB**

A multiple of a vector **u** is a vector k**u**, where **k** is a real number different from zero, and whose position is a multiple of the oriented abscissa k**AB**.

0**u** = 0**AB** = **AA** (zero oriented abscissa)

-1 **u** = -1 **AB** = **-u** (a reverse vector to vector **u**)

u = AB = B-A

-u = -AB = BA = A-B

Exercise: **u** [4,3], **v** [-2,-4]. Find the vectors:

- **a = u+v**

- **b = u-v**
- **c = 2u**
- **d = u + (-1/2) v**
- **e = 3/2 u – ½ v**

LINEAR COMBINATION

Let's have n-arbitrary vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \ldots, \mathbf{v_n}$. Each vector

$\mathbf{u} = k_1\mathbf{v_1} + k_2\mathbf{v_2} + \ldots + k_n\mathbf{v_n}$, where $k_1, k_2, \ldots, k_n$ are real numbers , is called a linear combination of vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \ldots, \mathbf{v_n}$.

E.g. vectors **a,b,c,d,e** are linear combinations of vectors **u** and **v.**

Ex: We have vectors **z** [2,10], **u** [1,3], **v** [-2,2]. Write the vector **z** as a linear combination of vectors **u, v.**

$\mathbf{z} = k_1 \mathbf{u} + k_2 \mathbf{v}$

$z_1 = k_1u_1 + k_2 v_1$

$z_2 = k_1u_2 + k_2v_2$

LINEAR DEPENDANCE / INDEPENDENCE

We say that vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \ldots, \mathbf{v_n}$ are linear dependent, if some of them can be expressed as a linear combination of the others. In other case we say that these vectors are linear independent.

Two vectors are linear dependent when their arbitrary positions are parallel. Such vectors are **collinear** (they may be *congruent* when they have the same direction, or *incongruent* when they are of the opposite direction).

If three vectors are linear dependent, then any of their positions lie in mutually parallel planes and such vectors are called **coplanar** vectors.

THE SCALAR PRODUCT

The **magnitude = size** of a vector is the size of any its position **AB**. I.e. it is the distance of the marginal points A, and B.

The size of zero vector is zero.

The unit vector is a vector whose size is 1.

The size of a k-multiple of a vector **u** is /kv/.

If we know the coordinates of a vector **u** = $[v_x, v_y, v_z]$ then the size of vector **u** is:

$$/\mathbf{u}/ = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

The angle (deviation) of two vectors

Let's have two vectors **u=AB, v=AC.** The angle of two non-zero vectors **u, v** is a convex angle BAC, for which it is valid that: φ = /<BAC/

$$0 \leq \varphi \leq 180\ (\pi)$$

<u>The scalar product</u> of ***two*** vectors **a** and **b** is defined as the product of the magnitudes of the two vectors multiplied with the cosine of the angle between them:

a b = /a/ /b/ cos φ

The term scalar product is used because /a/ /b/ and cos φ are all scalar quantities and so their product will be a scalar quantity as well.

We read **a b** as 'a dot b' and for this reason the scalar product may also be referred to as the **dot** product.

Note that the angle between two vectors always refers to the angle between the directions of the vectors when these directions are either both towards their point of intersection or both away from their point of intersection.

PROPERTIES OF THE SCALAR PRODUCT

1. From the definition it follows that two perpendicular vectors will have a scalar product of zero.
2. **a . a = /a/ /a/ cos 0 = /a/ /a/ = /a^2/**
3. The scalar product is commutative, i.e. **a . b = b . a**
4. The scalar product is distributive over addition, i.e. **a . (b+c) = a . b + a . c**
5. The properties for multiplication by a scalar λ are that:
   λ(**a . b) = a . (**λ**b) = (**λ**a) . b =** λ **/a/ . /b/** cos φ
6. If **a** = (a_1, a_2, a_3), **b** = (b_1, b_2, b_3), then the scalar product **a . b =** $a\, b_1 + a_2\, b_2 + a_3\, b_3$

VECTOR PRODUCT

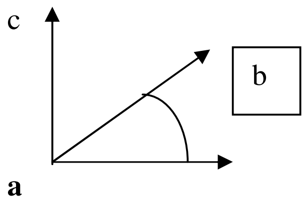
The vector product is possible only with vectors which lie in the space. The result of the vector product is a **vector**. The order of the vectors is important**: a** x **b** .

If at least one of the vectors **a, b** is a zero vector, then the vector product will be zero:

a x **b = 0.**

If none of the vectors is a zero vector, then the vector product is defined as: **c = a** x **b.**

1. **c** by the dextro- coordinate system has the direction assigned with the rule of the right hand used over the vectors **a, b** in this order



2. **c** is orthogonal on the vectors **a, b**
3. the magnitude = size of the vector **c** is /c/ = a x b = /a/ . /b/ . sin φ

THE PROPERTIES OF THE VECTOR PRODUCT

1. **a** x **b** = - (**b** x **a**) (anti-commutative)
2. if **a =** k . **b**, then **a** x **b** = **0** (the vector product of parallel vectors is zero)
3. likewise **a** x **a = 0** since the angle between **a** and **a** is zero
4. The vector product of perpendicular vectors – considering the unit vectors **i** and **j,** we have **i** x **j =** 1 x 1 sin 90 = 1. Vectors **i, j, k** form a right-handed set. Hence we have **i** x **j = k, j** x **k = i, k** x **i = j.**
   We notice that these vector products are **positive** when the alphabetical order in which **i, j** and **k** are taken is clockwise, but **negative** when this order is anticlockwise.
5. coordinates of the vector product **c = u** x **v**
   $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$
   $\mathbf{c} = \mathbf{u} \times \mathbf{v} = (u_2v_3 - v_2u_3; u_3v_1 - v_3u_1; u_1v_2 - v_1u_2)$

   $u_2 \; u_3 \; u_1 \; u_2$
   $v_2 \; v_3 \; v_1 \; v_2$

6. k . **u** x **v** = **u** x k . **v =** k . (**u** x **v**) (associative over the multiplication by the scalar)
7. (**u** + **v**) x **w = u** x **w + v** x **w** (distributive over addition)

APPLICATIONS

AREA OF A TRIANGLE ABC = ½ / **a** x **b/**

AREA OF A TETRAGON ABCD = / **a** x **b/**

MIXED PRODUCT = SCALAR TRIPLE PRODUCT

The scalar triple product of vectors **a, b** and **c** is defined as **(a x b) . c** . We must calculate the vector product first. A quick way to find the mixed product is to use determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

When **a, b** and **c** are **coplanar** (i.e. they lie in the same plane) their mixed product is **(a x b) . c = 0.**

APPLICATIONS

VOLUME OF A CUBOID V = /**(a x b) . c/**

VOLUME OF A PARALLELEPIPED V = /**(a x b) . c** /

(it is a polyhedron with six faces, each of which is a parallelogram)

VOLUME OF A TETRAHEDRON V = 1/6 /**(a x b) . c/**

(it is a polyhedron with four faces, each of which is a triangle. That is, a pyramid with a triangular base)

VOLUME OF A TRIANGULAR PRISM V = ½ /**(a x b) . c/**

VOLUME OF A PYRAMID V = 1 /3 /**(a x b) . c/**