SETS

A **set** is a file of objects which have at least one property in common. The objects of the set are called **elements**.

Sets are notated with capital letters K, Z, N, etc., the elements are a, b, c, d, etc. To specify the set we use curly brackets {…}.

We will use the symbol $a \in A$ to show that the element a belongs to the set A. On the other hand $a \notin A$ means that the element a does not belong to the set A.

The sets are determined in two ways:

- by enumerating elements, which belong to the set, e.g. A = {a, b, c}
- by stating characteristic properties of the objects from the set, e.g. $C = \{x \in N;\ x > 3\}$

Sets are of two sorts:

- finite
- infinite

A **system of sets** is a set, whose elements are again sets.

Set Notation

The most flexible way to write down objects belonging to a certain total is to use **set notation**.

Sets are delimited by curly braces. You can write down finite sets as lists.

For instance $\{-1, \sqrt{10}, \pi\}$ is the set with the three elements -1, π and $\sqrt{10}$.

For sets with infinitely many elements this becomes impossible, so there are other ways to write them down.

Special symbols are used to denote important sets:

- **N** is the set of natural numbers 1, 2, 3, ...
- **Z** denotes the integers 0, 1, -1, 2, -2, ...
- **Q** denotes the set of rational numbers (fractions).
- **R** denotes the set of all real numbers, consisting of all rational numbers and irrational numbers such as $\sqrt{10}$.
- **C** denotes the set of all complex numbers.
- Ø is the empty set, the set which has no elements.

Beyond that, set notation uses descriptions: the interval $(-3,5\rangle$ is written in set notation as

$\{x \in R; -3 < x \leq 5\}$read as " the set of all real numbers x such that $-3 < x \leq 5\}$."

The first part tells us what "universe" of numbers we are considering (in our case the universe of real numbers), the semicolon ";" separates the "universe" part from the second part, where we describe the property our numbers in the set are supposed to satisfy.

The set $\{x \in Z; -3 < x \leq 5\}$ is the set of all integers exceeding -3 and not greater than 5; this is a finite set; we could write it as a list,

$\{x \in Z; -3 < x \leq 5\} = \{-2, -1, 0, 1, 2, 3, 4, 5\}$

The set $\{x \in N; -3 < x \leq 5\}$ is even smaller; it contains only five elements:

$\{x \in N; -3 < x \leq 5\} - \{1, 2, 3, 4, 5\}$

Here are some more examples:

The interval $(3; \infty)$ can be written as $\{x \in R; x > 3\}$.

The set $(-\infty, 0) \cup (0, \infty)$ looks like this in set notation: $\{x \in R; x \neq 0\}$, or like $\{x \in R; x < 0 \text{ or } x > 0\}$.

OPERATIONS BETWEEN SETS

a) equality of sets
b) set inclusion (one set is included into another one)
c) union of sets
d) intersection of sets
e) difference of sets
f) complement of sets

EQUALITY OF SETS

Two sets A and B are equal, if each element from the set A is at the same time an element of the set B, and each element of B is an element of A.

The equality of sets has the following characteristics:

1. reflexive A = A (this is true for each set)
2. symmetric $A = B \Rightarrow B = A$
3. transitive $A = B; B = C \Rightarrow A = C$

SET INCLUSION

A set A is a ***subset*** of the set B then and only then if each element of A is at the same time an element of B ($A \subset B$). B is then called a hyper-set of A.

The set inclusion has the following characteristics:

1. reflexive $A \subset A$
2. anti-symmetric $A \subset B$ and $B \subset A \Leftrightarrow A = B$
3. transitive $A \subset B, B \subset C \Rightarrow A \subset C$

Each non-empty set has at least two subsets: itself and an empty set, e.g. if A = {2}, then its subsets are {2}, { }.

<u>Example</u>: Write all subsets of the set M = {a, b, c}

Subsets are: {a, b, c}, { }, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}

UNION OF SETS

A union of two sets is a set, which contains elements occurring in at least one of these sets.

$C = A \cup B = \{x;\ x \in A \vee x \in B\}$.

INTERSECTION OF SETS

An intersection of two sets is a set, which contains elements occurring in both sets at the same time.

$C = A \cap B = \{x;\ x \in A \wedge x \in B\}$.

If the intersection of the sets A and B is an empty set, then the sets A and B are called ***disjoint*** sets (mutually exclusive sets, non-overlapping sets).

Characteristics of union and intersection of sets

	UNION	INTERSECTION
1.	commutative $A \cup B = B \cup A$	commutative $A \cap B = B \cap A$
2.	associative $A \cup (B \cup C) = (A \cup B) \cup C$	associative $A \cap (B \cap C) = (A \cap B) \cap C$
3.	distributive in view of intersection $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$	distributive in view of union $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

DIFFERENCE OF SETS

A difference of sets is a set, which contains those elements of the set A, which do not belong to the set B. $A - B = \{x \in A,\ x \notin B\}$.

COMPLEMENT OF SETS

The basic set is a set which consists of all elements and all subsets we will work with.

A complement of a set A into the set X is the set, which contains those elements from X, which do not belong to A.

$A'_x = \{x \in X; x \notin A\}$

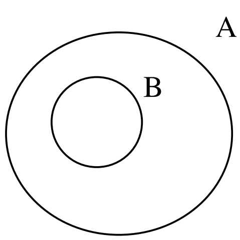
$A'_x = X - A$

De Morgan laws

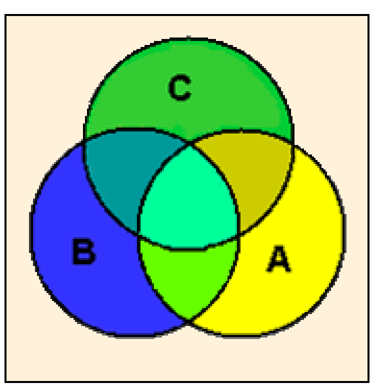
1. $(A \cap B)' = A' \cup B'$
2. $(A \cup B)' = A' \cap B'$

VENN DIAGRAMS

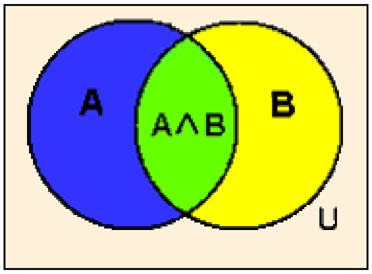
Venn diagrams are used to depict the sets and operations between sets. A Venn diagram is a drawing, in which circular areas represent groups of items sharing common properties. The drawing consists of two or more circles, each representing a specific group. This process of visualizing logical relationships was devised by John Venn (1834-1923).



$B \subset A$



Each Venn diagram begins with a rectangle representing the universal set. Then each set in the problem is represented by a circle. Any values that belong to more than one set will be placed in the sections where the circles overlap.



The Venn diagram at the left shows two sets **A** and **B**. Values that belong to both set **A** and set **B** are located in the centre region labelled **A** ∧ **B** where the circles overlap.

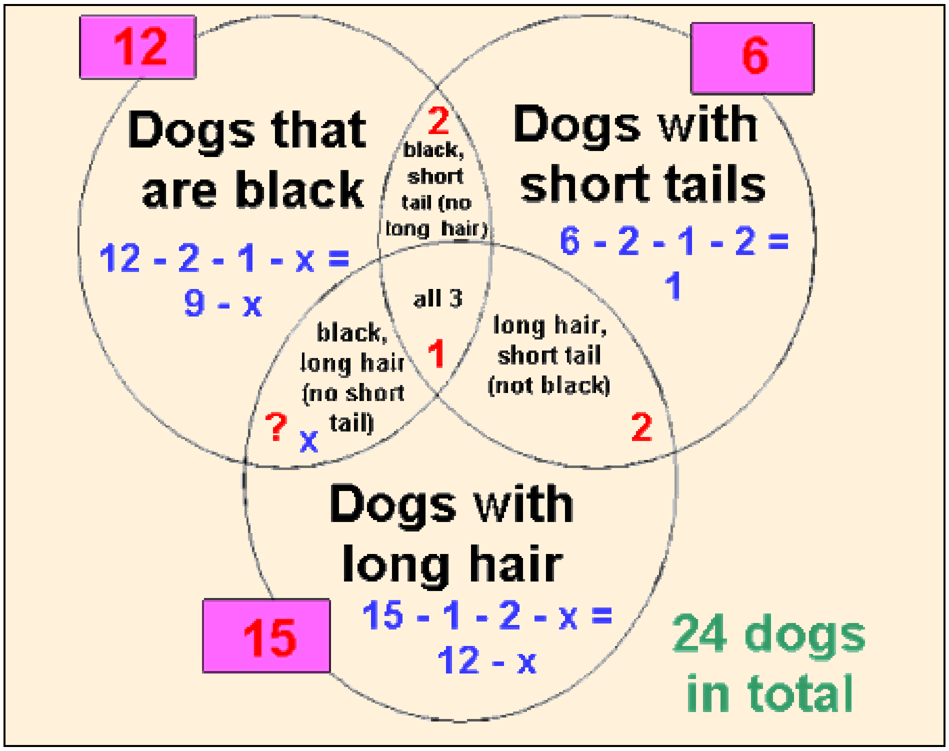
The notation **A v B** represents the entire region covered by both sets **A** and **B**.
If we cut out sets **A** and **B**, the remaining region in **U**
would be labelled **(A v B)’** and called complement.

Venn diagrams have the ability to represent a "sharing of conditions", which makes them useful tools for solving complicated problems. Consider the following example:

Example:

Twenty-four dogs are in a kennel. Twelve of the dogs are black, six of the dogs have short tails, and fifteen of the dogs have long hair. There is only one dog that is black with a short tail and long hair. Two of the dogs are black with short tails and do not have long hair. Two of the dogs have short tails and long hair but are not black. If all of the dogs in the kennel have at least one of the mentioned characteristics, how many dogs are black with long hair but do not have short tails?

Solution:
Draw a Venn diagram to represent the situation described in the problem.
Represent the number of dogs that you are looking for with x.



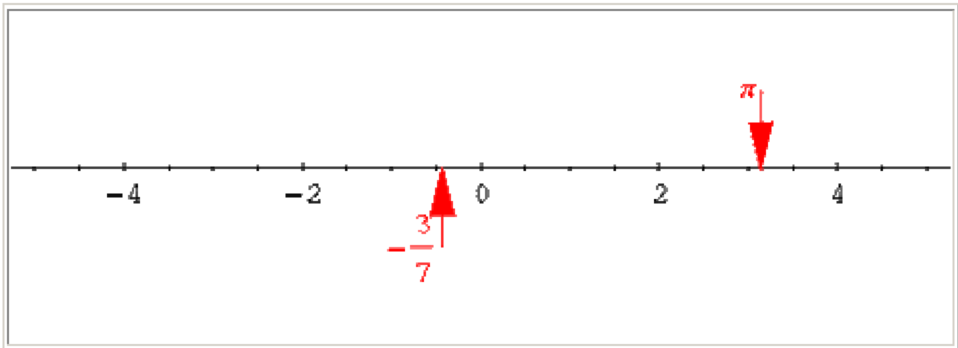
- Notice that the number of dogs in each of the three categories is labelled **OUTSIDE** of the circle in a coloured box. This number is a reminder of the total of the numbers which may appear anywhere inside that particular circle.
- After you have labelled all of the conditions listed in the problem, use this **OUTSIDE** box number to help you determine how many dogs are to be labelled in the empty sections of each circle.
- Once you have **EVERY** section in the diagram labelled with a number or an expression, you are ready to solve the problem.
- Add together **EVERY** section in the diagram and set it equal to the total number of dogs in the kennel (24). Do NOT use the **OUTSIDE** box numbers.

- $9 - x + 2 + 1 + 1 + 2 + x + 12 - x = 24$
- $27 - x = 24$
- $x = 3$ (There are 3 dogs which are black with long hair but do not have a short tail.)

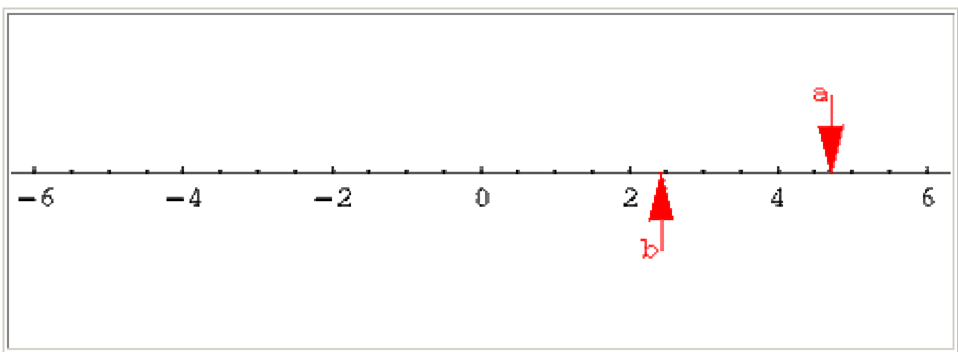
Venn diagrams are illustrations used in the branch of mathematics known as set theory. They show all of the possible mathematical and logical relationships between sets (groups of things).

The Real Number Line vs. Intervals

The most intuitive way to notate numbers is to use the **real number line**. If we draw a line, designate a point on the line to be zero, and choose a scale, then every point on the line corresponds uniquely to a real number, and vice versa:

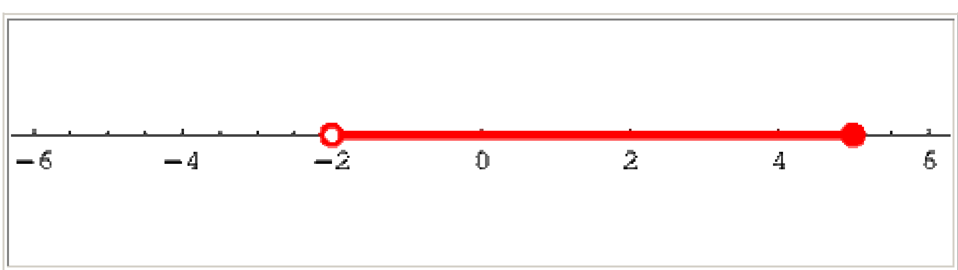


The real number line "respects" the order of the real numbers. A bigger number will always be found to the right of a smaller number. In the picture below, *a*>*b*.



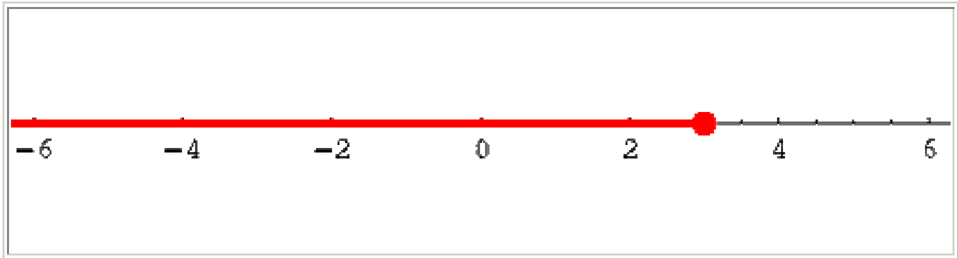
We visualize a set on the real number line by marking its members.

It is standard to agree on the following conventions: To **include** an endpoint, we "bubble it in." To **exclude** an endpoint, we use an "empty bubble". Here is the set of all real numbers greater than -2 and less than or equal to 5:

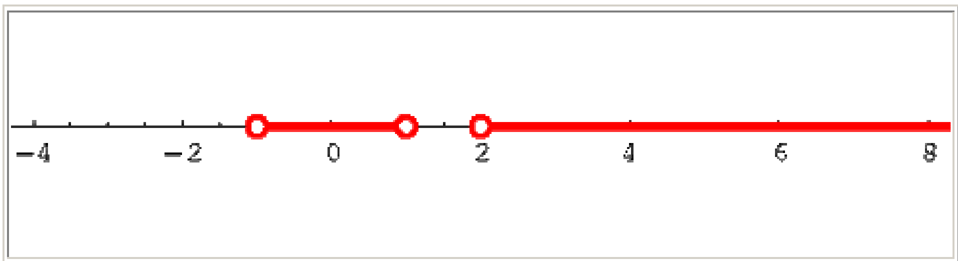


The number -2 is excluded from the set, so you see an "empty bubble"; the number 5 is included in the set, so the bubble at 5 is "filled in."

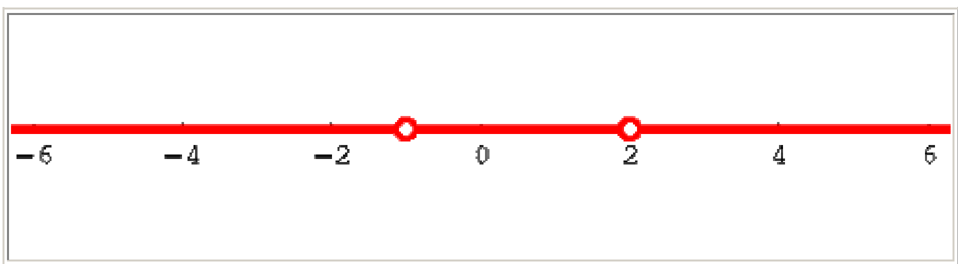
Next comes an unbounded set, the set of all numbers less than or equal to 3:



The set does not need to be "connected." The following graph depicts all real numbers which are either greater than 2 or strictly between -1 and 1.



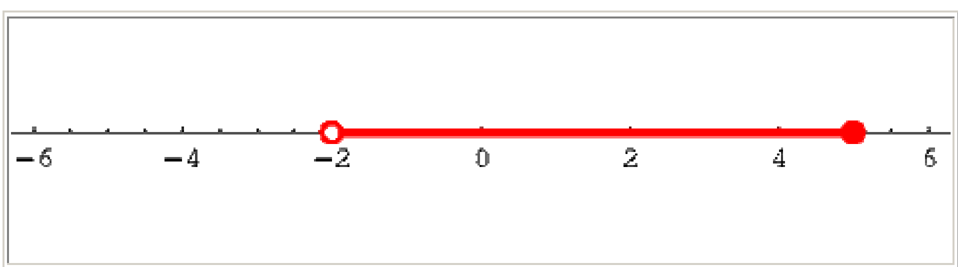
The following is a description of the set of all real numbers with the exception of -1 and 2:



Interval Notation

Interval notation translates the information from the real number line into symbols.

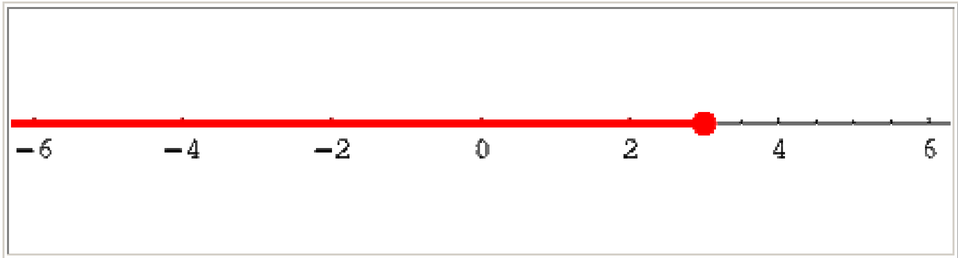
Our example



becomes the interval $(-2,5\rangle$.

To indicate that an endpoint is included, we use a pointed bracket; to exclude an endpoint, we use parentheses.

Our example

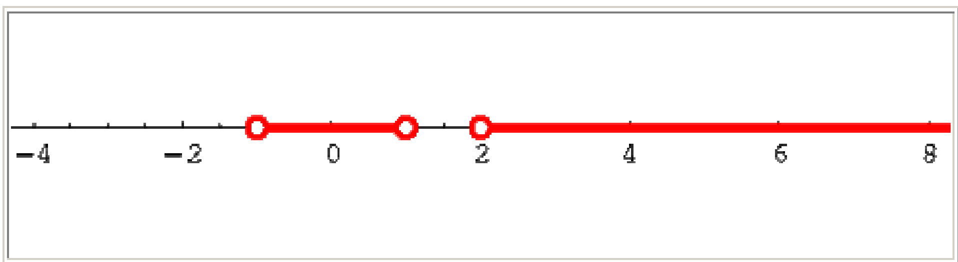


is written in interval notation as $(-\infty, 3\rangle$.

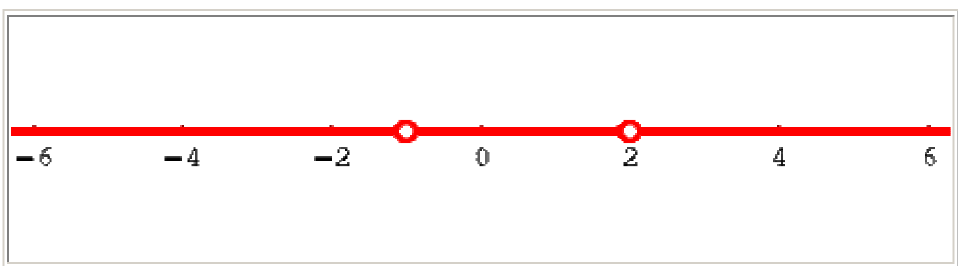
The infinity symbols "$+\infty$ " and " - ∞" are used to indicate that the set is unbounded in the positive ($+\infty$) or negative ($-\infty$) direction of the real number line. " $+\infty$" and " $-\infty$" are not real numbers, just symbols. Therefore we always exclude them as endpoints by using parentheses.

If the set consists of several disconnected pieces, we use the symbol for union " $\cup$":

Our example



is written in interval notation as (-1, 1) $\cup$ (2, ∞).



In the previous example, there are three pieces to consider:

$$(-\infty, -1) \cup (-1, 2) \cup (2, +\infty)$$

An interval such as (-1, $\sqrt{2}$) where both endpoints are excluded is called an **open interval**. Also the following unbounded interval (2, ∞) is considered to be open.

An interval is called **closed**, if it contains its endpoints, such as $\langle -\sqrt{3}, \pi \rangle$. An interval such as $(-\infty, 3\rangle$ is called closed (even though it does not contain its left endpoint).

The whole real line ($-\infty$, ∞) is considered to be both open and closed.

Intervals using both pointed and round brackets $\langle$ …) or (… $\rangle$ are called **half-closed intervals** or **half-open intervals**.

In algebra, an **interval** is a set that contains every real number between two indicated numbers and possibly the two numbers themselves. **Interval notation** is the notation in which permitted values for a variable are expressed as ranging over a certain interval; "$5 < x < 9$" is an example of the application of interval notation. In conventional interval notation, parentheses (...) indicate exclusion while pointed brackets $\langle ... \rangle$ indicate inclusion. For example, the interval (10,20) indicates the set of all real numbers between 10 and 20 but does *not* include 10 or 20, the first and last numbers of the interval, respectively. On the other hand, the interval $\langle 10;20 \rangle$ includes both every number between 10 and 20 *as well as* 10 and 20. Other possibilities are listed below.

Intervals of R are of the following eleven different types (where *a* and *b* are real numbers, with *a* < *b*):

1. $(a; b) = \{x; a < x < b)$
2. $\langle a;b \rangle = \{x; a \leq x \leq b)$
3. $\langle a; b) = \{x; a \leq x < b)$
4. $(a; b \rangle = \{x; a < x \leq b)$
5. $(a; \infty) = \{x; x > a\}$
6. $\langle a; \infty) = \{x; x \geq a\}$
7. $(-\infty, b) = \{x; x < b\}$
8. $(-\infty, b \rangle = \{x; x \leq b\}$
9. $(-\infty, \infty)$ = R itself, the set of all real numbers
10. $\varnothing$, the empty set
11. {a}, singleton

The last two are called **degenerate intervals**. In each case where they appear above, *a* and *b* are known as **endpoints** of the interval.
The endpoints will be separated by a semicolon to avoid ambiguity when the endpoints are decimal numbers.