Sequences and Series

A **sequence** is a special kind of function whose *domain* is N - the set of natural numbers. The *range* of a sequence is the collection of terms that make up the sequence. Just as the word *sequence* implies, the order of the terms in a sequence is important.

The first term of a sequence, for example, is found by taking the value of the function at 1; the second term is the value of the function at 2, and so on. Consider the sequence $f(x) = x$. The terms of the sequence, denoted a_1, a_2, a_3,…, a_n are 1, 2, 3,…, n. When working with sequences, instead of using function notation to express the formula of the function, a formula of the following form is used: $\boldsymbol{a_n = n}$. This is the same sequence as above, but the conventional n is used to denote a natural number, since only natural numbers are in the domain of sequences.

For graphical representation of sequences a *numerical line* or *coordinate system* are used.

Properties of sequences

Since a sequence is a special kind of function it has analogous properties to functions:

A sequence $\{a_n\}_{n=1}^{\infty}$ is **increasing** if and only if it is true that for every natural number n $a_{n+1} > a_n$, e.g. $\{\frac{n}{n+1}\}$.

A sequence $\{a_n\}_{n=1}^{\infty}$ is **decreasing** if and only if it is true that for every natural number n $a_{n+1} < a_n$, e.g. $\{\frac{1}{n+1}\}$.

A sequence $\{a_n\}_{n=1}^{\infty}$ is **non-decreasing** if and only if it is true that for every natural number n $a_{n+1} \geq a_n$.

A sequence $\{a_n\}_{n=1}^{\infty}$ is **non-increasing** if and only if it is true that for every natural number n $a_{n+1} \leq a_n$.

Increasing, decreasing, non-increasing and non-decreasing sequences are called **monotonic sequences**.

Of course, not every sequence has to be monotonic, e.g. $\{(-1)^n\}_{n=1}^{\infty}$ has these terms -1, 1, -1, 1, etc. and hence is not monotonic.

A sequence $\{a_n\}_{n=1}^{\infty}$ is **constant** if for all its terms it is true that $a_{n+1} = a_n$.

A sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded** from **below** if there exists such a number $b \in R$ that all its terms $a_n \geq b$, e.g. $\{\frac{n+1}{n}\}_{n=1}^{\infty}$.

A sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded** from **above** if there exists such a number $a \in R$ that all its terms $a_n \leq a$, e.g. $\{3 - n\}$.

A sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded** when it is bounded form *above and below* at the same time.

In other words, for all its terms a_n it is true that $b \leq a_n \leq a$, e.g. $\frac{1}{3} \leq \{\frac{n}{n+1}\}_{n=1}^{\infty} \leq 1$.

Two important categories of sequences are **arithmetic** sequences, and **geometric** sequences. Both are examples of a **recursive sequence** - a sequence in which *each term (besides the first) depends on the previous term.*

When the terms of a sequence are summed, the result is called a **series**.

A sequence is a number pattern in a definite order following a certain rule.

Examples of sequences:

1) 1, 2, 3, 4, 5, 6, 7, ... *add 1 to the preceding term*

2) 2, 4, 7, 11, 16, 23, 31. *add 2 to the preceding term, add 3 to the next term, etc*

3) 1, 1, 2, 3, 5, 8, 13, 21, 34,... *add the two preceding terms together*- this sequence is known as the ***Fibionacci sequence***, as discovered by Leonardo of Pisa. This sequence occurs in nature, and Leonardo of Pisa derived it by studying the mating patterns of rabbits.

A series is a sum of terms in a sequence.

Using the above sequences, we have the following series:

1) 1 + 2 + 3 + 4 + 5 + 6 + 7 +...

2) 2 + 4 + 7 + 11 + 16 + 23 + 31.

3) 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 +...

Sequences and series can be **finite** or **infinite**. A finite sequence/series is one that eventually comes to an end, like the second one in the examples above. Infinite sequences/series are those that continue indefinitely, such as the first in the example as well as the Fibionacci sequence.

A sequence is usually given

- with a **general notation** $\{a_n\}_{n=1}^{\infty}$. If we substitute a natural number n into the notation, we will get the value of corresponding term of that sequence.
- **Recurrently**, i.e. some term is given as well as a relation between other two or more terms.

Arithmetic & Geometric Progressions

Arithmetic and geometric progressions, commonly abbreviated to A.P. and G.P. respectively, are two forms of sequences. Their definitions are given later in this section. The applications of these sequences are more theoretical than practical, though the idea can be used to calculate values (distances, length, cost, etc) for practical situations whereby sequences in the form of A.P./G.P. are employed.

Arithmetic Progressions

An arithmetic progression is a sequence in which each term (except the first term) is obtained from the previous term by ***adding*** a constant known as the common ***difference***.

$$a_{n+1} = a_n + d$$

An arithmetic *series* is formed by the addition of the terms in an arithmetic progression.
In any arithmetic progression the *difference* of any two following terms is constant *d*.

The value of *difference* bears the information on the monotony of the progression:

- If $d > 0$, then AP is *increasing*
- If $d < 0$, then AP is *decreasing*
- If $d = 0$, then AP is *constant*

Let the first term of an A. P. be *a* and common difference *d*. Then,

General form of an A. P.:

$a_1 = a$,

$a_2 = a + d$,

$a_3 = a + 2d$, etc.

$a_{n+1} = a_n + d$

*n*th term of an A. P.:

$a_n = a_1 + (n - 1)d$

Sum of first *n* terms of an A. P.:

$s_n = \frac{n}{2} \cdot (a_1 + a_n)$, i.e. (first term + last term)

$s_n = \frac{n}{2} \cdot (a_1 + a_1 + (n - 1)d)$,

$s_n = \frac{n}{2} \cdot (2a_1 + (n - 1)d)$,

This idea was from the mathematician Carl Friedrich Gauss, who, as a young boy, stunned his teacher by adding up 1 + 2 + 3 + ... + 99 + 100 within a few minutes. Here's how he did it:

He counted 101 terms in the series, of which 50 is the middle term. He also realised that adding the first and last numbers, 1 and 100, gives, 101; and adding the second and second last numbers, 2 and 99, gives 101, as well as 3 + 98 = 101 and so on. Thus he concluded that there are 50 sets of 101 and the middle term is 50. So the sum of the series is:

50 (1 + 100) + 50 = 5050.

This can be rewritten as:

100/2 (1 + 100) + 50 = 5050 or

101/2 (1 + 100) = 5050

Examples on **Arithmetic Progressions**

1. The sum of the first 10 terms in an arithmetic progression is 50 and the sum of the next 10 terms is 250. Find the thirteenth term.

Solution:

$s_{10} = 10/2\ [2a + (10 - 1)\ d] = 50$

$5\ (2a + 9d) = 50$

$2a + 9d = 10$ .....(1)

Sum of first 20terms = $20/2\ [2a + (20 - 1)\ d] = 250 + 50$

$10\ (2a + 19d) = 300$

$2a + 19d = 30$ .....(2)

Solving (1) and (2) simultaneously,

$d = 2, a = -4$

13th term $= a + (13 - 1) d$

$= -4 + 12(2)$

$= 20$

2. The second term of an arithmetic progression is nine times the fifth term and the sum of the first eight terms is 56. Find

(i) the first term and common difference.

(ii) the least number of terms of the A. P. which must be taken for the sum to be negative.

Solution:

Second term = 9 (Fifth term)

$a_1 + d = 9 (a_1 + 4d)$

$8a + 35d = 0$ .....(1)

Sum to first eight terms:

$8/2 [2a + (8 - 1) d] = 56$

$8a + 28d = 56$ .....(2)

Solving (1) and (2) simultaneously,

$d = -8, a = 35$

Let the least number of terms be n.

$n/2 [2 (35) + (-8) (n - 1)] < 0$

$39n - 4n^2 < 0$

$n (39 - 4n) < 0$

$n > 9\ 3/4$ *since $n > 0$*

The least number of terms is 10.

3. The series $\lg x + \lg 2 + \lg x^2 + \lg 4 + \lg x^3 + \lg 8 + \ldots$ is an arithmetic progression. Show that the sum of first ten terms is 55 lg 2x.

Solution:

$\lg x + \lg 2 + \lg x^2 + \lg 4 + \lg x^3 + \lg 8 + \ldots$

= (lg x + lg 2) + 2 (lg x + lg 2) + 3 (lg x + lg 2) + ... *power law of logarithms*

This is an A. P.with

first term = lg x + lg 2

= lg 2x

common difference = lg x + lg 2

= lg 2x

Sum to first 10terms = 10/2 [2 lg 2x + (10 - 1) lg 2x]

= 5 (11 lg 2x)

= 55 lg 2x

Geometric Progressions

A geometric progression is a sequence in which each term (except the first term) is derived from the preceding term by the multiplication of a non-zero constant, which is the common **ratio**, called the **quotient** in Slovak literature.

$$a_{n+1} = a_n r$$

A geometric series is formed by the addition of the terms in a geometric progression.

Examples:

1) 3, 6, 9, 12, ... *first term 3, common ratio 3*

2) 4, -8, 16, -32, ... *first term 4, common ratio -2*

Let the first term be *a* and common ratio be *r*.

- General form of a G. P.:

$a_1 = a$,

$a_2 = ar$

$a_3 = ar^2$ etc.

$a_m = a_s r^{m-s}$

$a_{n+1} = a_1 r^n$

- *n*th term of a G. P.=

$a_n = a_1 r^{n-1}$

Sum to first n terms of a G. P.: $s_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$

Examples **on Geometric Progressions**

1. A geometric progression has positive terms. The sum of the first six terms is nine times the sum of the first three terms. The seventh term is 320. Find

(i) the common ratio

(ii) the first term

(iii) the sum of first five terms

Solution:

Sum of first 6 terms = 9 (Sum of first 3 terms)

$$\frac{a(1-r^6)}{1-r} = \frac{9a(1-r^3)}{1-r}$$

$1 - r^6 = 9(1 - r^3)$

$r^6 - 9r^3 + 8 = 0$

$(r^3 - 8)(r^3 - 1) = 0$

$r = 2$ or $r = 1$ *(rejected, since $r \neq 1$)*

Seventh term,

$a.2^{7-1} = 320$

$a.2^6 = 320$

$a = 5$

Sum to first five terms

$$S_5 = \frac{5(1-2^5)}{1-2}$$

$= 155$

2. A geometric progression has a positive common ratio, and the sixth term is $7\frac{9}{32}$. The sum of the first two terms and the sum of the third and fourth terms are in the ratio 4 : 9. Find the sum of the first six terms.

Solution:

(First term + Second term) / (Third term + Fourth term)

$$\frac{a+ar}{ar^2+ar^3}=\frac{4}{9}$$

$$\frac{a(1+r)}{ar^2(1+r)}=\frac{4}{9}$$

$r^2 = 9/4$ *since a is not 0, r is not 1*

$r = 3/2$ *since r >0*

Sixth term = $a(3/2)^5 = 7\frac{19}{32}$

$a = 1$

Sum of first six terms

$$=\frac{1[(3/2)^6-1]}{3/2-1}=20\frac{25}{32}$$

Sum to Infinity

The sum to infinity is a finite value the sum of the first n terms of a geometric series tends to when n tends to infinite. Sum to infinity only exists when a series is convergent.

Sum to infinity is given by the expression:

$$S=\frac{a}{1-r}$$

and only exists if :

$$|r|<1$$

Examples:

1. *In a geometric progression, the common ratio is -1/3, and the sum of the first three terms is 14/27. Find*

(i) the second negative term

(ii) the sum to infinity

Solution:

Sum of first three terms:

$$\frac{a(1-(-\frac{1}{3})^3)}{1-(-\frac{1}{3})} = \frac{14}{27}$$

a = 2/3

The second negative term is the fourth term:

$$\frac{2}{3}\left(-\frac{1}{3}\right)^3 = -\frac{2}{81}$$

Sum to infinity:

$$= \frac{\frac{2}{3}}{1-(-\frac{1}{3})}$$

= 1/2

More examples on Arithmetic & Geometric Progressions

1. If the first, third and thirteenth terms of an arithmetic progression are in geometric progression, and the sum of the fourth and seventh terms of this arithmetic progression is 40, find the first term and the (non-zero) common difference.

Let the first term of the arithmetic progression be *a* and common difference be *d*.

First term = a

Third term = a + (3 - 1) d = a + 2d

Thirteenth term = a + (13 - 1) d = a + 12d

Since the first, third and thirteenth terms are in geometric progression,

$$\frac{a}{a+2d} = \frac{a+2d}{a+12d}$$

$$(a + 2d)^2 = a (a + 12d)$$

$$a^2 + 4ad + 4d^2 = a^2 + 12ad$$

$$2ad - d^2 = 0 \qquad .....(1)$$

Fourth term + Seventh term = 40

$$a + (4 - 1)d + a (7 - 1)d = 40$$

$$2a + 9d = 40$$

$$2a = 40 - 9d \qquad .....(2)$$

substitute (2) into (1):

$$d (40 - 9d) - d^2 = 0$$

$$d (4 - d) = 0$$

$$d = 0 \text{ (rejected) or } d = 4$$

$$a = 2$$

Terms and Formulae

Terms

Arithmetic Sequence - A sequence in which each term is a constant amount greater or less than the previous term. In this type of sequence, $a_{n+1} = a_n + d$, where d is a constant.

Common Ratio - In a geometric sequence, the ratio r between each term and the previous term.

Convergent Series - A series whose limit as $n\to\infty$ is a real number.

Divergent Series - A series, which has no limit or whose limit as $n\to\infty$ is either ∞ or $-\infty$.

Explicit Formula - A formula for the nth term of a sequence of the form a_n = some function of n.

Finite Sequence - A sequence which is defined only for positive integers less than or equal to a certain given integer.

Finite Series - A series which is defined only for positive integers less than or equal to a certain given integer.

Geometric Sequence - A sequence in which the ratio between each term and the previous term is a constant ratio.

Index of Summation - The variable in the subscript of Σ. For

$\sum_{i=1}^{n}$

a_n, i is the index of summation.

Infinite Sequence - A sequence which is defined for all positive integers.

Infinite Series - A series which is defined for all positive integers.

Recursive Sequence - A sequence in which a general term is defined as a function of one or more of the preceding terms. A sequence is typically defined recursively by giving the first term, and the formula for any term a_{n+1} after the first term.

Sequence - A function which is defined for the positive integers, i.e. natural numbers

Series - A sequence in which the terms are summed, not just listed.

Summation Notation -

$\sum_{k=1}^{n}$

$a_n = a_1 + a_2 + a_3 + a_4 + ... + a_n$. The symbol Σ and its subscript and superscript are the components of summation notation.

Term - An element in the range of a sequence. A sequence is rarely represented by ordered pairs, but instead by a list of its terms.

Formulae

Limit of an Infinite Geometric Series	For a geometric sequence $a_n = a_1r^{n-1}$, where $-1 < r < 1$, the limit of the infinite geometric series $\sum_{n=1}^{\infty} a_1r^{n-1} = \frac{a_1}{1-r}$. This is the same as the sum of the infinite geometric sequence $a_n = a_1r^{n-1}$.
Sum of a Finite Arithmetic Sequence	The sum of the first n terms of the arithmetic sequence is $S_n = n(\frac{a_1+a_n}{2})$ or $S_n = \frac{n}{2}(a_1 + (n-1)d)$, where d is the difference between each term.
Sum of a Finite Geometric Sequence	For a geometric sequence $a_n = a_1r^{n-1}$, the sum of the first n terms is $S_n = a_1(\frac{1-r^n}{1-r})$