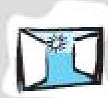
OPENING PROBLEM



At the 2004 IB Mathematics Teachers' Conference there were 273 delegates present. The organising committee consisted of 10 people.

- If each committee member shakes hands with every other committee member, how many handshakes take place?

  Can a 10-sided convex polygon be used to solve this problem?
- If all 273 delegates shake hands with all other delegates, how many handshakes take place now?

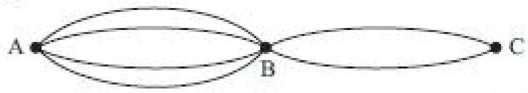
The opening problem is a counting problem. The following exercise helps us to count without actually listing and counting one by one. To do this we examine:

- the product principle
- counting permutations
- counting combinations

A THE PRODUCT PRINCIPLE

Suppose that there are three towns A, B and C and that 4 different roads could be taken from A to B and two different roads from B to C.

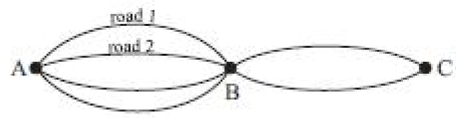
Diagrammatically we have:



The question arises: "How many different pathways are there from A to C going through B?"

If we take road 1, there are two alternative roads to complete our trip.

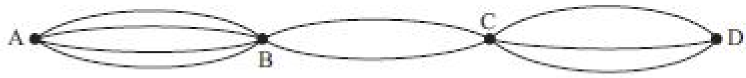
If we take road 2, there are two alternative roads to complete our trip. ...... etc.



So there are $2+2+2+2=4\times 2$ different pathways.

However, we notice that the 4 corresponds to the number of roads from A to B and the 2 corresponds to the number of roads from B to C.

Similarly, for



there would be $4\times 2\times 3=24$ different pathways from A to D passing through B and C.

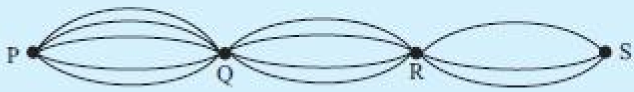
THE PRODUCT PRINCIPLE

The **product principle** is:

> If there are m different ways of performing an operation and for each of these there are n different ways of performing a second **independent** operation, then there are mn different ways of performing the two operations in succession."

The product principle can be extended to three or more successive operations.

Example 1



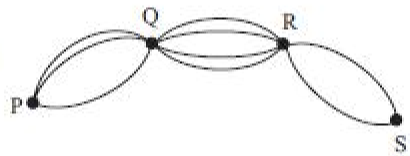
It is possible to take five different paths from Pauline's to Quinton's, 4 different paths from Quinton's to Reiko's and 3 different paths from Reiko's to Sam's. How many different pathways could be taken from Pauline's to Sam's via Quinton's and Reiko's?

The total number of different pathways $= 5 \times 4 \times 3 = 60$. {product principle}

EXERCISE 9A

1 The illustration shows the possible map routes for a bus service which goes from P to S through both Q and R.

How many different routes are possible?



2 

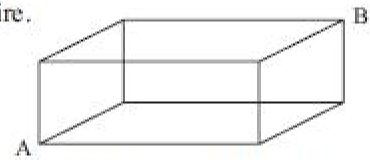
It is decided to label the vertices of a rectangle with the letters A, B, C and D.

In how many ways is this possible if:

- **a** they are to be in clockwise alphabetical order
- **b** they are to be in alphabetical order
- **c** they are to be in random order?

3 The figure alongside is box-shaped and made of wire. An ant crawls along the wire from A to B.

How many different paths of shortest length lead from A to B?



4 In how many different ways can the top two positions be filled in a table tennis competition of 7 teams?

5 A football competition is organised between 8 teams. In how many ways is it possible to fill the top 4 places in order of premiership points obtained?

6 How many 3-digit numbers can be formed using the digits 2, 3, 4, 5 and 6:

a as often as desired **b** once only?

7 How many different alpha-numeric plates for motor car registration can be made if the first 3 places are English alphabet letters and those remaining are 3 digits from 0 to 9?

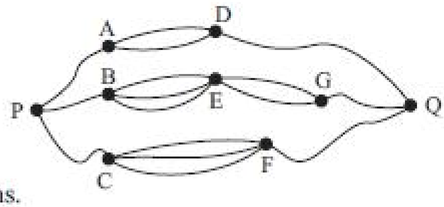
8 In how many ways can:

a 2 letters be mailed into 2 mail boxes **b** 2 letters be mailed into 3 mail boxes

c 4 letters be mailed into 3 mail boxes?

B COUNTING PATHS

Consider the following road system leading from P to Q:



From A to Q there are 2 paths.
From B to Q there are $3 \times 2 = 6$ paths.
From C to Q there are 3 paths.

Thus, from P to Q there are $2 + 6 + 3 = 11$ paths.

Notice that:

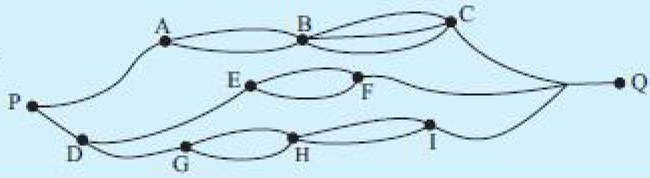
- when going from B to G, we go from B to E **and** then from E to G, and we multiply the possibilities,
- when going from P to Q, we must first go from P to A, **or** P to B **or** P to C, and we add the possibilities.

Consequently:

- the word **and** suggests multiplying the possibilities
- the word **or** suggests adding the possibilities.

Example 2

How many different paths lead from P to Q?



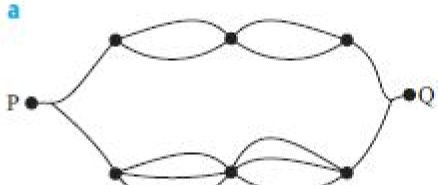
We could go from P to A to B to C to Q where there are $2 \times 3 = 6$ paths
or from P to D to E to F to Q where there are 2 paths
or from P to D to G to H to I to Q where there are $2 \times 2 = 4$ paths.

So, we have $6 + 2 + 4 = 12$ different alternatives.

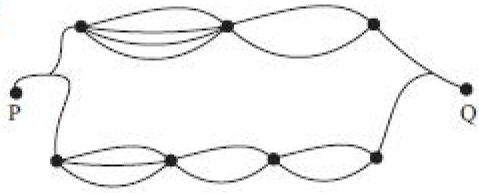
EXERCISE 9B

1 How many different paths lead from P to Q?

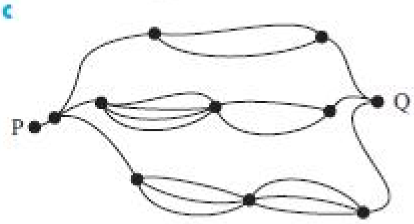
a



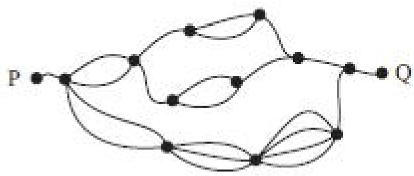
b



c



d



C FACTORIAL NOTATION

In problems involving counting, products and consecutive positive integers are common.

For example, $8 \times 7 \times 6$ or $6 \times 5 \times 4 \times 3 \times 2 \times 1$, etc.

FACTORIAL NOTATION

For convenience, we introduce **factorial numbers**, where numbers such as $6\times5\times4\times3\times2\times1$ are written as $6!$.

In general, $n!$ is the product of the first n positive integers for $n \geqslant 1$
i.e., $n! = n(n-1)(n-2)(n-3) .... \times 3 \times 2 \times 1$, for $n \geqslant 1$
and $n! = 1$ for $n = 0$.
$n!$ is read "n factorial".

Notice that $8 \times 7 \times 6$ can be written using factorial numbers only as

$$8 \times 7 \times 6 = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{8!}{5!}$$

PROPERTIES OF FACTORIAL NUMBERS

The **factorial rule** is $n! = n \times (n-1)!$

which can be extended to $n! = n(n-1)(n-2)!$, etc.

Notice that, although 0! cannot be included in the original definition of factorial numbers, we can now give it a value.

Using the factorial rule with $n = 1$, we have $1! = 1 \times 0!$ i.e., $1 = 0!$

So we define $0! = 1$, and this is consistent with $n! = n \times (n-1)!$

Example 3

What integer is equal to: **a** $4!$ **b** $\frac{5!}{3!}$ **c** $\frac{7!}{4! \times 3!}$?

a $4! = 4 \times 3 \times 2 \times 1 = 24$

b $\frac{5!}{3!} = \frac{5 \times 4 \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2 \times 1}} = 5 \times 4 = 20$

c $\frac{7!}{4! \times 3!} = \frac{7 \times 6 \times 5 \times \cancel{4 \times 3 \times 2 \times 1}}{\cancel{4 \times 3 \times 2 \times 1} \times 3 \times 2 \times 1} = 35$

EXERCISE 9C

1 Find $n!$ for $n = 0, 1, 2, 3, ..., 10$.

2 Simplify without using a calculator:

a $\frac{6!}{5!}$ **b** $\frac{6!}{4!}$ **c** $\frac{6!}{7!}$ **d** $\frac{4!}{6!}$ **e** $\frac{100!}{99!}$ **f** $\frac{7!}{5! \times 2!}$

3 Simplify: **a** $\frac{n!}{(n-1)!}$ **b** $\frac{(n+2)!}{n!}$ **c** $\frac{(n+1)!}{(n-1)!}$

Example 4

Express in factorial form: **a** $10\times9\times8\times7$ **b** $\dfrac{10\times9\times8\times7}{4\times3\times2\times1}$

a $10\times9\times8\times7=\dfrac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{6\times5\times4\times3\times2\times1}=\dfrac{10!}{6!}$

b $\dfrac{10\times9\times8\times7}{4\times3\times2\times1}=\dfrac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1\times6\times5\times4\times3\times2\times1}=\dfrac{10!}{4!\times6!}$

4 Express in factorial form:

a $7\times6\times5$ **b** 10×9 **c** $11\times10\times9\times8\times7$

d $\dfrac{13\times12\times11}{3\times2\times1}$ **e** $\dfrac{1}{6\times5\times4}$ **f** $\dfrac{4\times3\times2\times1}{20\times19\times18\times17}$

Example 5

Write the following sums/differences as a product by factorising:

a $8!+6!$ **b** $10!-9!+8!$

a $8!+6!$
$=8\times7\times6!\ +\ 6!$
$=6!(8\times7+1)$
$=6!\times57$

b $10!-9!+8!$
$=10\times9\times8!\ -\ 9\times8!+8!$
$=8!(90-9+1)$
$=8!\times82$

5 Write as a product (using factorisation):

a $5!+4!$ **b** $11!-10!$ **c** $6!+8!$ **d** $12!-10!$

e $9!+8!+7!$ **f** $7!-6!+8!$ **g** $12!\ -\ 2\times11!$ **h** $3\times9!\ +\ 5\times8!$

Example 6

Simplify $\dfrac{7!-6!}{6}$ using factorisation.

$\dfrac{7!-6!}{6}=\dfrac{7\times6!-6!}{6}$

$=\dfrac{6!(7-1)}{6}$

$=6!$

6 Simplify using factorisation:

a $\dfrac{12!-11!}{11}$ **b** $\dfrac{10!+9!}{11}$ **c** $\dfrac{10!-8!}{89}$ **d** $\dfrac{10!-9!}{9!}$

e $\dfrac{6!+5!-4!}{4!}$ **f** $\dfrac{n!+(n-1)!}{(n-1)!}$ **g** $\dfrac{n!-(n-1)!}{n-1}$ **h** $\dfrac{(n+2)!+(n+1)!}{n+3}$

D COUNTING PERMUTATIONS

A **permutation** of a group of symbols is *any arrangement* of those symbols in a definite *order*.

For example, BAC is a permutation on the symbols A, B and C when all three of them are used, i.e., taken 3 at a time.

Notice that ABC, ACB, BAC, BCA, CAB, CBA are all the different permutations on the symbols A, B and C taken 3 at a time.

In this exercise we are concerned with listings of all permutations, and counting how many permutations there are, without having to list them all.

Example 7

List all the permutations on the symbols P, Q and R when they are taken:
a 1 at a time **b** 2 at a time **c** 3 at a time.

a P, Q, R

b PQ QP RP
PR QR RQ

c PQR QPR RPQ
PRQ QRP RQP

Example 8

List all permutations on the symbols W, X, Y and Z taken 4 at a time.

WXYZ	WXZY	WYXZ	WYZX	WZXY	WZYX
XWYZ	XWZY	XYWZ	XYZW	XZYW	XZWY
YWXZ	YWZX	YXWZ	YXZW	YZWX	YZXW
ZWXY	ZWYX	ZXWY	ZXYW	ZYWX	ZYXW

i.e., 24 of them.

For large numbers of symbols listing the complete set of permutations is absurd. However, we can still count them in the following way.

Consider **Example 8** again:

There are 4 positions to fill

1st	2nd	3rd	4th

Into the 1st position, any of the 4 symbols could be used.

This leaves any 3 symbols to go into the 2nd position, which in turn leaves any 2 symbols to go into the 3rd position, and finally leaves the remaining 1 symbol to go into the 4th position.

Consequently,

1st	2nd	3rd	4th
4	3	2	1

and so the total number $= 4 \times 3 \times 2 \times 1$ {product principle}
$= 24$

EXERCISE 9D

1 List the set of all permutations on the symbols W, X, Y and Z taken
a 1 at a time **b** two at a time **c** three at a time.

(**Note:** **Example 8** has them taken 4 at a time.)

2 List the set of all permutations on the symbols A, B, C, D and E taken:

a 2 at a time **b** 3 at a time.

Example 9

If a chess association has 16 teams, in how many different ways could the top 8 positions be filled on the competition ladder?

Any of the 16 teams could fill the 'top' position.
Any of the remaining 15 teams could fill the 2nd position.
Any of the remaining 14 teams could fill the 3rd position.
⋮
Any of the remaining 9 teams could fill the 8th position.

i.e.,

16	15	14	13	12	11	10	9
1st	2nd	3rd	4th	5th	6th	7th	8th

$\therefore$ total number $= 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9$
$= 518\,918\,400$

3 In how many ways can:

a 5 different books be arranged on a shelf
b 3 different paintings, from a collection of 8, be hung in a row
c a signal consisting of 4 coloured flags be made if there are 10 different flags to choose from?

4 Suppose you have 4 different coloured flags. How many different signals could you make using:

a 2 flags only **b** 3 flags only **c** 2 or 3 flags?

Example 10

You have available the alphabet blocks A, B, C, D and E and they are placed in a row. For example you could have: [D|A|E|C|B]

a How many different permutations could you have?
b How many permutations end in C?
c How many permutations have form [...|A|...|B|...] ?
d How many begin and end with a vowel, i.e., A or E?

a There are 5 letters taken 5 at a time.

$\therefore$ total number $= 5 \times 4 \times 3 \times 2 \times 1 = 120$.

b [4|3|2|1|1]

any others here

C here

C goes into the last position (i.e., 1 way) and the other 4 letters could go into the remaining 4 places in 4! ways.

$\therefore$ total number $= 1 \times 4! = 24$ ways.

c | 3 | 1 | 2 | 1 | 1 |
↑ A ↑ B

A goes into 1 place, B goes into 1 place and the remaining 3 letters go into the remaining 3 places in 3! ways.

$\therefore$ total number $= 1 \times 1 \times 3! = 6$ ways.

d | 2 | 3 | 2 | 1 | 1 |
↑ A or E ↑ remainder of A or E

A or E could go into the 1st position, and after that one is placed, the other one goes into the last position.

The remaining 3 could be arranged in 3! ways in the 3 remaining positions.

$\therefore$ total number $= 2 \times 1 \times 3! = 12$.

5 How many different permutations of the letters A, B, C, D, E and F are there if each letter can be used once only? How many of these:

a end in ED **b** begin with F and end with A
c begin and end with a vowel (i.e., A or E)?

6 How many 3-digit numbers can be constructed from digits 1, 2, 3, 4, 5, 6 and 7 if each digit may be used:

a as often as desired **b** only once **c** once only and the number is odd?

7 In how many ways can 3 boys and 3 girls be arranged in a row of 6 seats? In how many of these ways do the boys and girls alternate?

8 Numbers of 3 different digits are constructed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 using a digit once only. How many such numbers:

a can be constructed **b** end in 5 **c** end in 0 **d** are divisible by 5?

Example 11

There are 6 different books arranged in a row on a shelf. In how many ways can two of the books, A and B be together?

Method 1: We could have any of the following locations for A and B

A	B	×	×	×	×
B	A	×	×	×	×
×	A	B	×	×	×
×	B	A	×	×	×
×	×	A	B	×	×
×	×	B	A	×	×
×	×	×	A	B	×
×	×	×	B	A	×
×	×	×	×	A	B
×	×	×	×	B	A

} 10 of these

If we consider any one of these, the remaining 4 books could be placed in 4! different orderings

$\therefore$ total number of ways $= 10 \times 4! = 240$.

Method 2:

A and B can be put together in 2! ways (i.e., AB or BA).
Now consider this pairing as one symbol (tie a string around them) which together with the other 4 books (i.e., 5 symbols) can be ordered in 5! different ways.

$\therefore$ total number $= 2! \times 5! = 240$.

9 In how many ways can 5 different books be arranged on a shelf if:

- **a** there are no restrictions
- **b** books X and Y must be together
- **c** books X and Y are never together?

10 A group of 10 students randomly sit in a row of 10 chairs. In how many ways can this be done if:

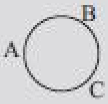
- **a** there are no restrictions
- **b** 3 students A, B and C are always seated together?

INVESTIGATION 1 PERMUTATIONS IN A CIRCLE

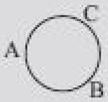
There are 6 permutations on the symbol A, B and C **in a line**.

These are: ABC ACB BAC BCA CAB CBA.

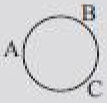
However **in a circle** there are only 2 different permutations on these 3 symbols. These are:

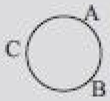


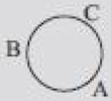
and



as they are the only possibilities with different right-hand and left-hand neighbours.







are the same cyclic permutations.

What to do:

1 Draw diagrams showing different cyclic permutations for:

- **a** one symbol; A
- **b** two symbols; A and B
- **c** three symbols; A, B and C
- **d** four symbols; A, B, C and D

2 Copy and complete:

Number of symbols	*Permutations in a line*	*Permutations in a circle*
1		
2		
3	$6 = 3!$	$2 = 2!$
4		

3 If there are n symbols to be permuted in a circle, how many different orderings are possible?