Limit

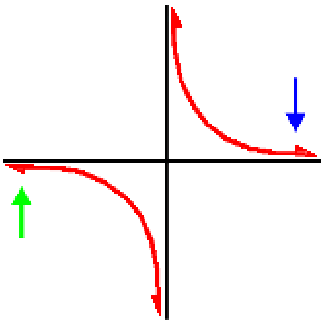
In mathematics, the concept of a "**limit**" is used to describe

- the behaviour of a function as its argument either gets "close" to some point, or as it becomes arbitrarily large;
- or the behaviour of a sequence's elements, as their index increases indefinitely.

Limits are used in calculus and other branches of mathematical analysis to define derivatives and continuity.

A GRAPHICAL EXAMPLE:

Now, let's look at the graph of $f(x)=\frac{1}{x}$ and see what happens!



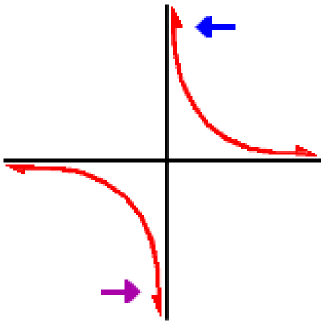
The x-axis is a horizontal asymptote... Let's look at the blue arrow first. As x gets really, really big, the graph gets closer and closer to the x-axis which has a height of 0. So, as x approaches infinity, f(x) is approaching 0. This is called a *limit at infinity*.

$$\lim_{x \to +\infty} \left(\frac{1}{x} \right) = 0$$

Now let's look at the green arrow... What is happening to the graph as x gets really, really small? The graph is again getting closer and closer to the x-axis (which is 0.) It's just coming in from below this time.

$$\lim_{x \to -\infty} \left(\frac{1}{x} \right) = 0$$

But what happens as x approaches 0?



Since different things happen, we need to look at two separate cases: what happens as x approaches 0 from the left *and* what happens as x approaches 0 from the right:

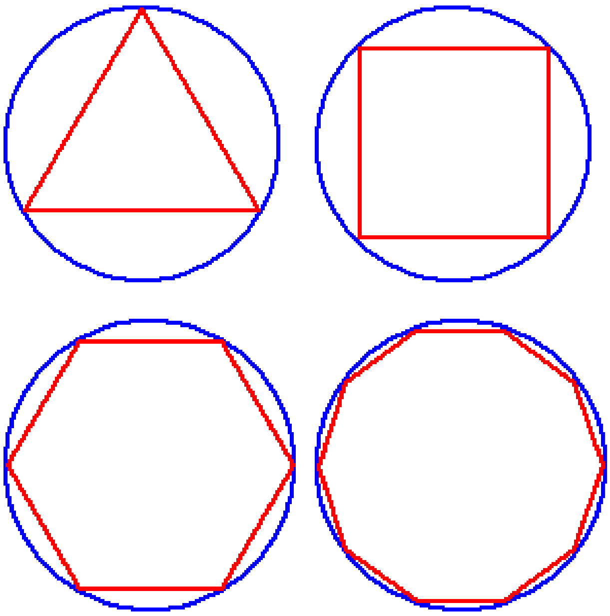
$$\lim_{x \to 0^-}\left(\frac{1}{x}\right) = -\infty \quad \text{and} \quad \lim_{x \to 0^+}\left(\frac{1}{x}\right) = \infty$$

Since the limit from the left does *not* equal the limit from the right...

$$\lim_{x \to 0}\left(\frac{1}{x}\right) = \text{does not exist}$$

A GEOMETRIC EXAMPLE:

Let's look at a polygon inscribed in a circle... If we increase the number of sides of the polygon, what can you say about the polygon with respect to the circle?



As the number of sides of the polygon increases, the polygon is getting closer and closer to the circle!

If we refer to the polygon as an *n-gon*, where n is the number of sides, we can make some equivalent mathematical statements.

As n gets larger, the n-gon gets closer to being the circle.

As n *approaches* infinity, the n-gon *approaches* the circle.

The *limit* of the n-gon, as n goes to infinity, is the circle!

$$\lim_{n \to \infty} (\text{n-gon}) = \text{circle}$$

The n-gon never really gets to be the circle, but it will get pretty close! So close, in fact, that, for all practical purposes, it may as well be the circle. That's what limits are all about!

Archimedes used this idea to find the area of a circle before they had a value for PI! (They knew PI was the circumference divided by the diameter... But, they didn't have calculators back then.)

Limit of a sequence

Consider the following sequence: 1.79, 1.799, 1.7999,... We could observe that the numbers are "approaching" 1.8, the limit of the sequence.

Formally, suppose $x_1, x_2, \ldots$ is a sequence of real numbers. We say that the real number L is the *limit* of this sequence and we write

$$\lim_{n \to \infty} x_n = L$$

if and only if

for every real number $\varepsilon > 0$ there exists a natural number n_0 (which will depend on ε) such that for all $n > n_0$ we have $| x_n - L | < \varepsilon$.

Intuitively, this means that eventually all elements of the sequence get as close as we want to the limit, since the absolute value $| x_n - L |$ can be interpreted as the "distance" between x_n and L. Not every sequence has a limit; if it does, we call it ***convergent***, otherwise ***divergent***. One can show that **a convergent sequence has only one limit.**

SOME NUMERICAL EXAMPLES:

EXAMPLE 1:

Let's look at the sequence whose n^{th} term is given by $\frac{n}{n+1}$. Recall, that we let n = 1 to get the first term of the sequence, we let n = 2 to get the second term of the sequence and so on.

What will this sequence look like?

1/2, 2/3, 3/4, 4/5, 5/6,... 10/11,... 99/100,... 99999/100000,...

What's happening to the terms of this sequence? Can you think of a number that these terms are getting closer and closer to? The terms are getting closer to 1! But, will they ever get to 1? No! So, we can say that these terms are approaching 1. It sounds like a limit! The limit is 1.

As n gets bigger and bigger, $\frac{n}{n+1}$ gets closer and closer to 1...

$$\lim_{n \to \infty} \left(\frac{n}{n+1} \right) = 1$$

EXAMPLE 2:

Now, let's look at the sequence whose n^{th} term is given by $\frac{1}{n}$. What will this sequence look like?

1/1, 1/2, 1/3, 1/4, 1/5,... 1/10,... 1/1000,... 1/1000000000,...

As n gets bigger, what are these terms approaching? That's right! They are approaching 0.

How can we write this in Calculus language?

$$\lim_{n \to \infty} \left(\frac{1}{n} \right) = 0$$

Examples:

1. $\lim_{n\to\infty} \frac{n(4n-2)}{(5-n)(3+n)} = \lim_{x\to\infty} \frac{4n^2-2n}{15+2n-n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{x\to\infty} \frac{4-\frac{2}{n}}{\frac{15}{n^2}+\frac{2}{n}-1} = \lim_{x\to\infty} \frac{4}{-1} = -4$

2. $\lim_{n\to\infty} \frac{2n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{2}{n}}{1+\frac{1}{n^2}} = \frac{0}{1+0} = \underline{\underline{0}}$

Limit of a function

Suppose $f(x)$ is a real function and c is a real number. The expression:

$$\lim_{x\to c} f(x) = L$$

means that $f(x)$ can be made to be as close to L as desired by making x sufficiently close to c. In that case, we say that "the limit of f of x, as x approaches c, is L". Note that this statement can be true even if $f(c) \neq L$. Indeed, the function $f(x)$ need not even be defined at c. Two examples help illustrate this.

Consider $f(x) = \frac{x}{x^2+1}$ as x approaches 2. In this case, $f(x)$ is defined at 2 and equals its limit of 0.4:

f(1.9)	f(1.99)	f(1.999)	f(2)	f(2.001)	f(2.01)	f(2.1)
0.4121	0.4012	0.4001	⇒0.4 ⇐	0.3998	0.3988	0.3882

As x approaches 2, $f(x)$ approaches 0.4 and hence we have $\lim_{x\to 2} f(x) = 0.4$.

In the case where $f(c) = \lim_{x\to c} f(x)$, f is said to be continuous at $x = c$. But it is not always the case.

Consider the case where $f(x)$ is undefined at $x = c$.

$$f(x) = \frac{x-1}{\sqrt{x}-1}$$

In this case, as x approaches 1, $f(x)$ is undefined at $x = 1$ but the limit equals 2:

f(0.9)	f(0.99)	f(0.999)	f(1.0)	f(1.001)	f(1.01)	f(1.1)
1.95	1.99	1.999	⇒undef ⇐	2.001	2.010	2.10

Thus, x can get as close to 1, so long as it is not equal to 1, so that the limit of $f(x)$ is 2.

FORMAL DEFINITION

A limit is formally defined as follows: Let f(x) be a function defined on an open interval containing c (except possibly at c) and let L be a real number.

$$\lim_{x\to c} f(x) = L$$

means that

for each real $\epsilon > 0$ there exists a real $\delta > 0$ such that for all x where $0 < |x - c| < \delta$, $|f(x) - L| < \epsilon$.

In words, $\lim_{x \to c} f(x) = L$ if and only if by taking *x* close enough to *c* we can get *f*(*x*) arbitrarily close to *L*.

The formal definition of a limit is sometimes called the ***delta-epsilon form*** because it uses the Greek letters delta (δ) and epsilon (ε). In practice, δ and ε are just variables and could be replaced by any other letters.

Properties of the Limit

Each of the following properties is proven using the rigorous definition of the limit. Let **lim** stand for $\lim_{x \to c}$, $\lim_{x \to c^+}$, or $\lim_{x \to c^-}$

Assume **lim f(x)** and **lim g(x)** both exist.

- **(Uniqueness)** If lim f(x) = L_1 and lim f(x) = L_2, then $L_1 = L_2$.
- **(Addition)** lim [f(x) + g(x)] = lim f(x) + lim g(x)
- **(Scalar multiplication)** lim [c . f(x)] = c . lim f(x).
- **(Multiplication)** lim [f(x) . g(x)] = = lim f(x) . lim g(x).
- **(Division)** lim $\frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$, provided lim g(x) $\neq 0$.
- **(Powers)** lim $[f(x)]^n$ = $[\lim f(x)]^n$ for any positive integer n.

In practice, much of the time we can ``reason out'' the value of a limit without explicitly using the ε-δ definition.

LIMITS OF FUNCTIONS AS X APPROACHES A CONSTANT

The following problems require the use of the algebraic computation of limits of functions as x approaches a constant. Most problems are average. A few are somewhat challenging. All of the solutions are given WITHOUT the use of L'Hopital's Rule. If you are going to try these problems before looking at the solutions, you can avoid common mistakes by giving careful consideration to the form $\frac{0}{0}$ during the computations of these limits. Initially, many students INCORRECTLY conclude that $\frac{0}{0}$ is equal to 1 or 0 , or that the limit does not exist or is ∞ or - ∞. In fact, the form $\frac{0}{0}$ is an example of an *indeterminate* form. This simply means that you have *not yet determined* an answer. Usually, this indeterminate form can be circumvented by using algebraic manipulation. Such tools as algebraic simplification, factorising, and conjugates can easily be used to circumvent the form $\frac{0}{0}$ so that the limit can be calculated.

As a result of these theorems, we see that for many functions f,

$$\lim_{x \to a} f(x) = f(a)$$

A function which has this property is called continuous. Polynomial functions and rational functions are continuous.

The following examples demonstrate how we can evaluate limits of functions which are not continuous by using the above-mentioned list of limit theorems.

Examples:

1. $$\lim_{x \to 3} \frac{5x^2 - 8x - 13}{x^2 - 5} = \frac{5(3)^2 - 8(3) - 13}{(3)^2 - 5} = \frac{8}{4} = 2$$.

2. $$\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 - 4} = \text{"}\frac{0}{0}\text{"}$$

(Circumvent the indeterminate form by factoring both the numerator and denominator.)

$$= \lim_{x \to 2} \frac{(x-2)(3x+5)}{(x-2)(x+2)}$$

(Divide out the factors x - 2 , the factors which are causing the indeterminate form $\frac{0}{0}$.

Now the limit can be computed.)

$$= \lim_{x\to 2} \frac{(3x+5)}{(x+2)} = \frac{3(2)+5}{(2)+2} = \frac{11}{4}$$

3.

$$\lim_{x\to 4} \frac{3-\sqrt{x+5}}{x-4} = "\frac{0}{0}"$$

(Eliminate the square root term by multiplying by the conjugate of the numerator over itself. Recall that (a – b) (a + b) = $a^2 - b^2$

$$= \lim_{x\to 4} \frac{3-\sqrt{x+5}}{x-4}\,\frac{3+\sqrt{x+5}}{3+\sqrt{x+5}} = \lim_{x\to 4} \frac{9-(x+5)}{(x-4)(3+\sqrt{x+5})}$$

$$= \lim_{x\to 4} \frac{4-x}{(x-4)(3+\sqrt{x+5})} = \lim_{x\to 4} \frac{-(x-4)}{(x-4)(3+\sqrt{x+5})}$$

(Divide out the factors x - 4 , the factors which are causing the indeterminate form $\frac{0}{0}$.

Now the limit can be computed.)

$$= \lim_{x\to 4} \frac{-1}{3+\sqrt{x+5}} = \frac{-1}{3+\sqrt{4+5}} = -\frac{1}{6}.$$

4.

$$\lim_{x\to 0} \frac{sin(5x)}{3x} = "\frac{0}{0}"$$

(If you wrote that sin (5x) = 5 sin x, you are incorrect. Instead, multiply and divide by 5, since we want to apply the well-known fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

$$= \lim_{x\to 0} \frac{5}{5}\,\frac{sin(5x)}{3x} = \lim_{x\to 0} \frac{5}{3}\,\frac{sin(5x)}{5x} = \frac{5}{3}(1) = \frac{5}{3}.$$

Limit of a function at infinity

A related concept to limits as *x* approaches some finite number is the limit as *x* approaches *positive or negative infinity*. This does not literally mean that the difference between *x* and infinity becomes small, since infinity is not a real number; rather, it means that *x* either grows without bound positively (positive infinity) or grows without bound negatively (negative infinity).

For example, consider $f(x) = \frac{2x}{x+1}$.

$f(100) = 1.9802$

$f(1000) = 1.9980$

$f(10000) = 1.9998$

As *x* becomes extremely large, the value of $f(x)$ approaches 2, and the value of $f(x)$ can be made as close to 2 as one could wish just by picking *x* sufficiently large. In this case, we say that the limit of $f(x)$ as *x* approaches infinity is 2. In mathematical notation,

$$\lim_{x\to\infty} f(x) = 2.$$

Formally, we have the **definition**

$\lim_{x\to\infty} f(x) = c$ if and only if for each $\varepsilon > 0$ there exists an *n* such that $| f(x) - c | < \varepsilon$ whenever $x > n$.

Note that the *n* in the definition will generally depend on ε. A similar definition applies for $\lim_{x\to-\infty} f(x) = c$.

If one considers the domain of f to be the extended real number line, then the limit of a function at infinity can be considered as a special case of limit of a function at a point.