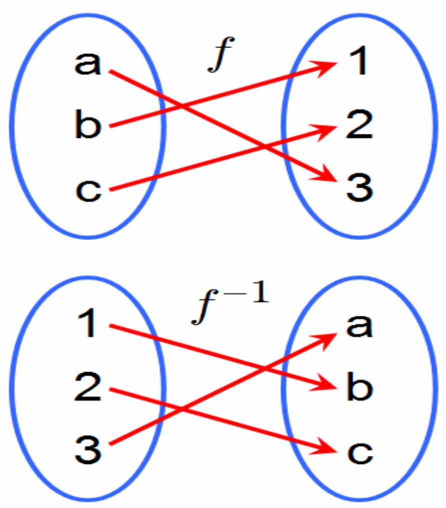
INVERSE FUNCTION

In mathematics, if f is a function from A to B, then an **inverse function** f^{-1} for f is a function in the opposite direction, from B to A, with the property that a round trip (a composition) from A to B to A and/or from B to A to B returns each element of the initial set to itself. Not every function has an inverse; those that do are called **invertible**.



Basically speaking, the process of finding an ***inverse*** is simply the swapping of the x and y coordinates. This newly formed inverse will be a relation, but may not necessarily be a ***function.*** It is also true that the ***inverse of a function*** may not necessarily form another ***function.***

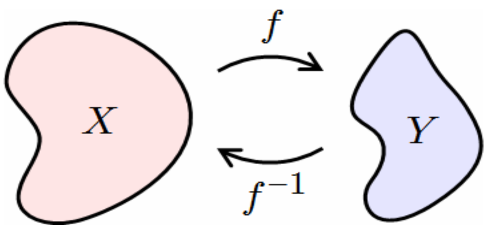
Remember: **The inverse of a function may not always be a *function*!**
The original function must be a *one-to-one function* to guarantee that its inverse will also be a function.

Let f be a function whose domain is the set X, and whose range is the set Y. Then the **inverse** of f is the function f^{-1} with domain Y and range X, defined by the following rule:

$$\text{If } f(x) = y, \text{ then } f^{-1}(y) = x.$$

Thus, an inverse function uniquely identifies the input x of another function based only on its output y, for all $y \in Y$. Not all functions have an inverse. For this rule to be appliable, each element $y \in Y$ must correspond to exactly one element $x \in X$. A function f with this property is called one-to-one, or an injection.

For instance, if $f(x) = y = x^2$, each element in Y would correspond to two different elements in X ($\pm x$), and therefore f would not be invertible. More precisely, the square of x is not invertible because it is impossible to deduce from its output the sign of its input. Such a function is called non-injective.



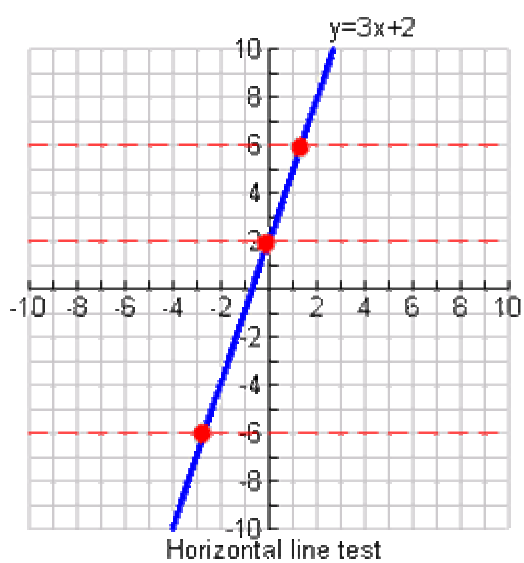
Uniqueness - If an inverse function exists for a given function f, it is unique.

For f to be invertible, it must be injective or bijective. It is shares the property of symmetry which can be concisely expressed by the following formula:

$$\left(f^{-1}\right)^{-1} = f.$$

Definition: A function is a ***one-to-one function*** if and only if each second element corresponds to one and only one first element. (each x and y value is used only once)

Use the **horizontal line test** to determine if a function is a *one-to-one function*.
If ANY horizontal line intersects your original function in ONLY ONE location, your function will be a one-to-one function and its inverse will also be a *function*.



The function y = 3x + 2, shown at the right, IS a one-to-one function and its inverse will also be a *function*.

(Remember that the **vertical line test** is used to show that a relation is a function.)

Definition: For all one-to-one functions, the ***inverse function*** is the set of ordered pairs obtained by interchanging the first and second elements of each ordered pair in the original function, i.e. (x, y) of *f* changes into (y, x) of f^{-1}.

Notation: If f is a given function, then f^{-1} denotes the inverse of f.

Swap ordered pairs: If your function is defined as a list of ordered pairs, simply swap the *x* and *y* values. Remember, the inverse will be a *function* only if the original function is ***one-to-one***.

Examples:

a. Given function *f*, find the inverse. Is the inverse also a *function*?:

$$f(x) = \{(3,4),(1,-2),(5,-1),(0,2)\}$$

Answer:

Function *f* is a one-to-one function since the *x* and *y* values are used only once. The inverse is

$$f^{-1}(x) = \{(4,3),(-2,1),(-1,5),(2,0)\}$$

Since function *f* is a one-to-one function, the inverse is also a function.

b. Determine the inverse of this function. Is the inverse also a *function*?

x	1	-2	-1	0	2	3	4	-3
$f(x)$	2	0	3	-1	1	-2	5	1

Answser: Swap the x and y variables to create the inverse. Since function f was **not** a one-to-one function (the y value of 1 was used twice), the inverse will **NOT** be a function (because the x value of 1 now gets mapped to two separate y values which is not possible for functions).

x	2	0	3	-1	1	-2	5	1
$f^{-1}(x)$	1	-2	-1	0	2	3	4	-3

Solve algebraically: Solving for an inverse algebraically is a three step process:

1. Set the function = y
2. Swap the x and y variables
3. Solve for y

Examples:

a. Find the inverse of the function $f(x) = x - 4$

Answer:

$y = x - 4$	*Remember:*
$x = y - 4$	Set = y.
$x + 4 = y$	Swap the variables.
$f^{-1}(x) = x + 4$	Solve for y.

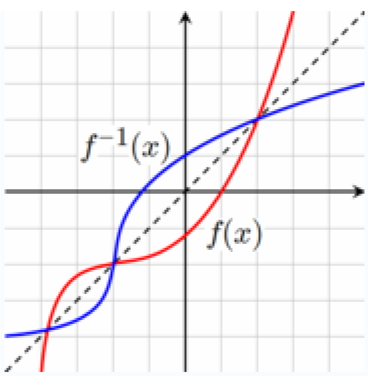
b.

Find the inverse of the function $f(x)=\dfrac{x+1}{x}$

Answer:

$y=\dfrac{x+1}{x}$	*Remember:* Set = *y*.
$x=\dfrac{y+1}{y}$	Swap the variables.
$xy=y+1$	
$xy-y=1$	
$y(x-1)=1$	Eliminate the fraction by multiplying each side by *y*.
$y=\dfrac{1}{x-1}$	Get the *y*'s on one side of the equal sign by subtracting *y* from each side.
$f^{-1}(x)=\dfrac{1}{x-1}$	Isolate the *y* by factoring out the *y*.
	Solve for *y*.

Graph of the inverse: The ***graph of an inverse*** is the ***reflection*** of the original graph over the identity line $y = x$. It may be necessary to restrict the domain on certain *functions* to guarantee that the inverse is also a *function*.



The graph of $y = f(x)$ is red and $y = f^{-1}(x)$ is blue. The dotted line is $y = x$.

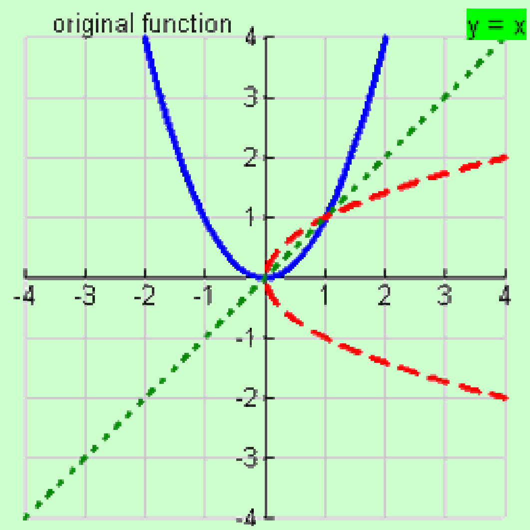
If f and f^{-1} are inverses, then the graph of the function

$y=f^{-1}(x)$ is the same as the graph of the equation $x=f(y)$.

This is identical to the equation $y = f(x)$ that defines the graph of f, except that the roles of x and y have been reversed. Thus the graph of f^{-1} can be obtained from the graph of f by switching the positions of the x and y axes. This is equivalent to reflecting the graph across the line $y = x$.

Example:

Consider, as our original function: $y = x^2$.
This original function is denoted in **blue**.



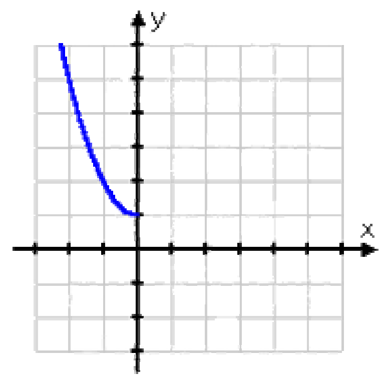
If reflected over the identity line $y = x$, the original function becomes the **red** dashed graph. Since the **red** graph will not pass the vertical line test for functions, our original function, $y = x^2$, does **not** have an inverse *function*. You can see that the inverse exists, but it is NOT a function.

With functions such as $y = x^2$, it is possible to **restrict the domain** to obtain an inverse *function* for a portion of the graph. This means that you will be looking at only a selected section of the original graph that **will** pass the horizontal line test for the existence of an inverse *function*. For example, restrict such as:

$y = x^2$ where $x \geq 0$ or $y = x^2$ where $x \leq 0$	}	by restricting the graph in such a manner, you guarantee the existence of an inverse *function* for a portion of the graph. (Other restrictions are also possible.)

Find the inverse function of $y = x^2 + 1$, $x \leq 0$.

The only difference between this function and the previous one is that *the domain has been restricted to only the negative half of the x-axis*. It looks like this:



With the domain restricted like this, the function ***has*** an inverse that is also a function. Just about any time they give you a problem where they've taken the trouble to restrict the domain, you should take care with the algebra and draw a nice picture, because the inverse probably is a function, but it will probably take some extra effort to show this. In this case, since the domain is $x \leq 0$ and the range (from the graph) is $1 \leq y$, then the inverse will have a domain of $1 \leq x$ and a range of $y \leq 0$. Here's how the algebra looks:

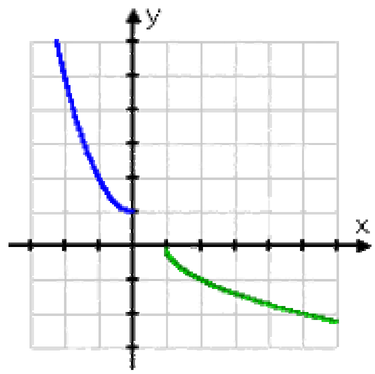
The original function:	$y = x^2 + 1, \quad x \leq 0$
Solve for "$x =$":	$y = x^2 + 1$
	$y - 1 = x^2$
	$\pm\sqrt{y-1} = x$
By figuring out the domain and range of the inverse, I know that I should choose the negative sign for the square root:	$-\sqrt{y-1} = x$

Switch the x and y; the "$y =$" is the inverse:	$y = -\sqrt{x-1}, \quad x \geq 1$

(The "$x \geq 1$" restriction comes from the fact that x is inside a square root.)

So **the inverse is $y = -\sqrt{x-1}$, $x \geq 1$, and this inverse is also a function.**

Here's the graph:



Sometimes the inverse of a function cannot be expressed by a formula. For example, if f is the function

$$f(x) = x + \sin x,$$

then f is one-to-one, and therefore possesses an inverse function f^{-1}. There is no simple formula for this inverse, since the equation $y = x + \sin x$ cannot be solved algebraically for x.

It is important to realize that $f^{-1}(x)$ is not the same as $f(x)^{-1}$, the latter is referred to the numeric inverse.