Cartesian product

In mathematics, the **Cartesian product** is a direct product of sets. The Cartesian product is named after René Descartes, whose formulation of analytic geometry gave rise to this concept.

Specifically, the Cartesian product of two sets X (for example the points on an x-axis) and Y (for example the points on a y-axis), denoted $X \times Y$, is the set of all possible ***ordered pairs*** (x, y) whose first component is a member of X (i.e. $x \in X$) and whose second component is a member of Y (i.e. $y \in Y$). We will notate it as follows:

$X \times Y = \{(x, y);\ x \in X \text{ and } y \in Y\}$.

We can also construct the set of all possible functions from set X to set Y, which we denote by either $[X \rightarrow Y]$.

An ***ordered pair*** of components is a pair, for which the order is important.

Example:

A = {a; b; c}

B = {1; 2}

A × B = {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}

B × A = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}

The ***graph*** of a Cartesian product $X \times Y$ is the ***set of isolated points***.

Binary relation

In mathematics, a **binary relation** from A into B is each subset of the Cartesian product A × B.

A binary relation R is usually defined as an ordered triple (X, Y, G) where X and Y are arbitrary sets and G is a subset of the Cartesian product $X \times Y$. The sets X and Y are called the **domain** and **co-domain**, respectively, of the relation, and G is called its **graph**.

The statement $(x,y) \in R$ is read "*x* **is *R*-related to** *y*", and is denoted by xRy or $R(x,y)$, where R is the notation for the binary relation.

The all-important concept of function is defined as a special kind of binary relation.

The order of the elements in each pair of G is important: if $a \neq b$, then aRb and bRa can be true or false, independently of each other.

Special types of binary relations

Some important classes of binary relations R over X and Y are listed below

- **left-total**: for all x in X there exists a y in Y such that xRy
- **surjective** or right-total: for all y in Y there exists an x in X such that xRy.
- **functional** (also called right-definite): for all x in X, and y and z in Y it holds that if xRy and xRz then $y = z$.
- **injective**: for all x and z in X and y in Y it holds that if xRy and zRy then $x = z$.
- **bijective**: left-total, right-total, functional, and injective.

A binary relation that is both left-total and functional is called a **function**.

Relations over a set

If $X = Y$ then we simply say that the binary relation is over X. Or it is an **endo-relation** over X.

Some important classes of binary relations over a set X are:

- **reflexive**: for all x in X it holds that xRx. For example, "greater than or equal to: $\geq$" is a reflexive relation but "greater than: >" is not.
- **irreflexive**: for all x in X it holds that **not** xRx. "Greater than: >" is an example of an irreflexive relation.
- **coreflexive**: for all x and y in X it holds that if xRy then $x = y$.
- **symmetric**: for all x and y in X it holds that if xRy then yRx. "Is a blood relative of" is a symmetric relation, because x is a blood relative of y if and only if y is a blood relative of x.
- **antisymmetric**: for all x and y in X it holds that if xRy and yRx then $x = y$. "Greater than or equal to" is an antisymmetric relation, because if $x \geq y$ and $y \geq x$, then $x = y$.
- **transitive**: for all x, y and z in X it holds that if xRy and yRz then xRz. "Is an ancestor of" is a transitive relation, because if x is an ancestor of y and y is an ancestor of z, then x is an ancestor of z.

A relation which is *reflexive, symmetric* and *transitive* is called an ***equivalence relation***.
A relation which is *reflexive, antisymmetric* and *transitive* is called a ***partial order***.

Operations on binary relations

If R is a binary relation over X and Y, then the following is a binary relation over Y and X:

Inverse or **converse**: R^{-1}, defined as $R^{-1} = \{(y, x) \mid (x, y) \in R\}$.

A binary relation over a set is equal to its inverse if and only if it is symmetric.

The Image

The **image** Z from A to B is every *binary relation* from A to B, if it is true that to **every** $x \in A$ there exists **at most** 1 $y \in B$ so that the ordered pair (x, y) belongs to this binary relation Z.

Z is an image $\Leftrightarrow Z \subset A \times B$ and $[\forall x \in A\ \exists\ y \in B; (x, y) \in Z]$

The Domain

The domain D (Z) is a set of those elements $x \in A$, to which there exists $y \in B$ so that the ordered pair $(x, y) \in Z$.

$D(Z) = \{x \in A\ \ \exists\ y \in B; (x, y) \in Z\}$

The Range

The range H (Z) is a set of those elements $y \in B$, to which there exists $x \in A$ so that the ordered pair $(x, y) \in Z$.

$H(Z) = \{y \in B\ \exists\ x \in A;\ (x, y) \in Z\}$

Let's have the Cartesian product $A \times B$. If

- D (Z) = A, then the image is *injective* and is called 'from A into B'
- H (Z) = B, then the image is *surjective* and is called 'from A onto B'
- D (Z) = A, and H (Z) = B, then the image is *bijective* and is called 'A onto B'

FUNCTION

The mathematical concept of **function** expresses dependence between two quantities, one of which is given (the independent variable, argument of the function, or its "input", usually x) and the other produced (the dependent variable, value of the function, or "output", usually y).

A function associates a single output with **every** input element drawn from a fixed set, such as the real numbers.

There are many ways to give a function: by a formula, by a plot or graph, by an algorithm that computes it, by a description of its properties. Sometimes, a function is described through its relationship to other functions. In applied disciplines functions are frequently specified by their tables of values, or by a formula.

Mathematical functions are frequently denoted by letters, and the standard notation for the output of a function f with the input x is $y = f(x)$.

A function may be defined only for certain inputs, and the collection of all acceptable *inputs* of the function is called its **domain**. The set of all *resulting outputs* is called the **range** of the function.

Three definitions of a **function** follow:

- ✓ A function is given by an arithmetic expression describing how one number depends on another.
- ✓ A function is every image defined on the set of real numbers – such a function is called *a real function of a real variable* defined on the set $M \subset R$, and it is a set of ordered pairs (x, y) from the Cartesian product $M \times R$, for which it is true that to each $x \in M$ there exists exactly 1 $y \in R$.

  $\{(x, y) \in M \times R, \forall x \in M \; \exists!\, y \in R\}$
- ✓ A function f from a set X to a set Y associates to each element x in X an element $y = f(x)$ in Y.

The notation $f\colon X \to Y$ indicates that f is a function with domain X and range Y.

A specific input x in a function is called an **argument** of the function. For each argument value x, the corresponding unique y in the range is called the **function value** at x, or the **image** of x **under** f. The image of x may be written as $f(x)$ or as y.

The **graph** of a function f is the set of all ordered pairs $(x, f(x))$, for all x in the domain X. If X and Y are subsets of **R**, the real numbers, then this definition coincides with the familiar sense of "graph" as a picture or plot of the function, with the ordered pairs being the Cartesian coordinates of points.

The concept of the *image* can be extended from the image of a point to the image of a set. If A is any subset of the domain, then $f(A)$ is the subset of the range consisting of all images of elements of A. We say the $f(A)$ is the **image** of A under f.
Notice that the range of f is the image $f(X)$ of its domain.

There are the following ways of notating a function:

- Prescription rule, i.e. a mathematical expression
- Chart of ordered pairs
- Enumeration of elements
- Verbal description
- Graph

Substitution

When a function is represented algebraically, we are given the rule as it applies to some variable. This is called functional notation. To compute the rule applied to any input we simply ***replace*** the variable with the input.

Example 1:

Given: $f(x) = x^2$ then

$f(5) = (5)^2 = 25$

$f(-1) = (-1)^2 = 1$

$f(a + b) = (a + b)^2 = a^2 + 2ab + b^2$

$f(2y) = (2y)^2 = 4y^2$

Example 2:

Given: $h(x) = \frac{1}{x}$ then

h(1) = 1

h(3) = $\frac{1}{3}$

In this example, h(x) = $\frac{1}{x}$, so h(0) *doesn't make sense* since we *can't divide by zero*. When the function doesn't make sense for a particular input value, we say that the ***function is not defined for that input value***.

Three important kinds of function are

- ◊ the **injections** (or **one-to-one functions**), which have the property that if $f(a) = f(b)$ then a must equal b;
- ◊ the **surjections** (or **onto functions**), which have the property that for every y in the range there is an x in the domain such that $f(x) = y$;
- ◊ the **bijections**, which are both one-to-one and onto. This nomenclature was introduced by the Bourbaki group.

Formal description of a function typically involves the function's name, its domain, its range, and a rule of correspondence. Thus we frequently see a two-part notation, e.g.

f: N → R

f: n → $\frac{n}{\pi}$

where the first part is read:

"f is a function from **N** to **R**" (one often writes informally "Let f: $X \to Y$" to mean "Let f be a function from N into T"), and the second part is read: …. maps to …

Here the function named "f" has the natural numbers as domain, the real numbers as range, and maps n to itself divided by π.

If f is a function from X to Y then an **inverse function** for f, denoted by f^{-1}, is a function in the opposite direction, from Y to X, with the property that a round trip (a composition) returns each element to itself. Not every function has an inverse; those that do are called **invertible**.

As a simple example, if f converts a temperature in degrees Celsius to degrees Fahrenheit, the function converting degrees Fahrenheit to degrees Celsius would be a suitable f^{-1}.

In mathematics, **injections**, **surjections** and **bijections** are classes of functions distinguished by the manner in which *arguments* (input expressions from the domain) and *images* (output expressions from the range) are related or *mapped to* each other.

A function is **bijective** (**one-to-one and onto**) if and only if (iff) it is *both* injective and surjective. (Equivalently, *every* element of the range is mapped to by *exactly one* element of the domain.) A bijective function is a **bijection** (**one-to-one correspondence**).
A function is **bijective** if it is both injective and surjective. A bijective function is a **bijection** (**one-one correspondence**). A function is bijective if and only if every possible image is mapped to by exactly one argument.
The function is bijective iff for all arguments and images, there is a unique equation such that $f(a) = b$.

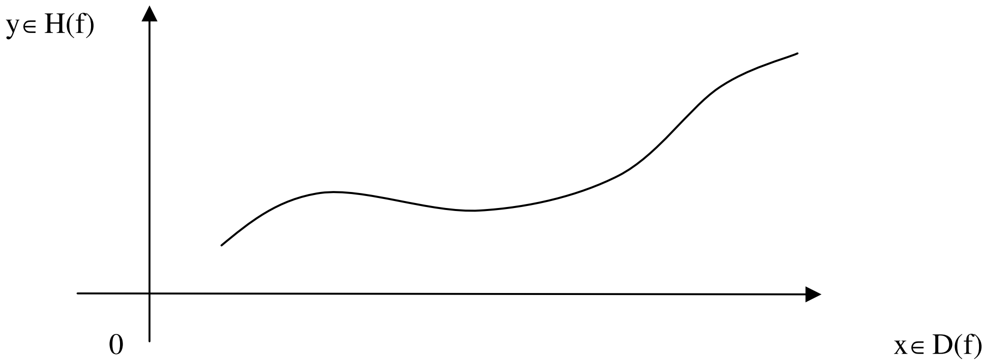
A function $f : A \rightarrow B$ is bijective if and only if it is invertible, that is, there is a function g: $B \rightarrow A$ such that $g \circ f$ = identity function on A and $f \circ g$ = identity function on B. This function maps each image to its unique preimage.
(*Note: a* ***one-to-one*** *function is injective, but may fail to be surjective, while a* ***one-to-one correspondence*** *is both injective and surjective.*)

An injective function need not be surjective (not all elements of the range may be associated with arguments), and a surjective function need not be injective (some images may be associated with *more than one* argument). There are four possible combinations of injective and surjective features:

- ◊ Injective and surjective (bijective)
- ◊ Injective and non-surjective
- ◊ Non-injective and surjective
- ◊ Non-injective and non-surjective

Graph

Set of all points X [x, f(x)], $x \in D(f)$ is called a **graph** of a function *f* in the coordinate system 0xy in plane.



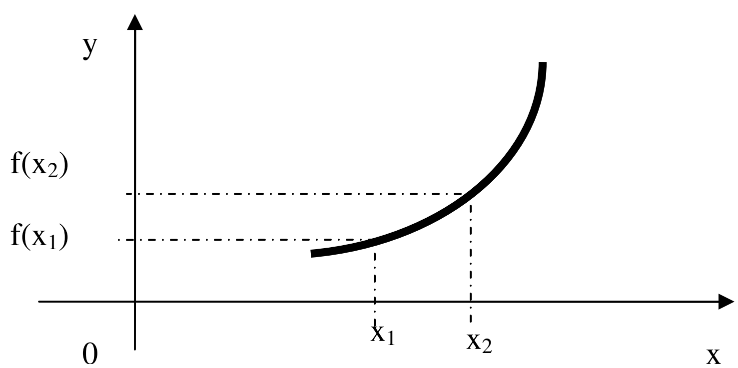
A graph is basically a picture of a function that can tell us, at a glance, lots of useful information about the function.

What we need is a picture that will tell us the output values y corresponding to different input values x. To do this, we need some machinery – rule = function.

MONOTONY

Let f (x) be a function and M be a subset of domain of this function.

Function *f* is an ***increasing function*** on set M if and only if for each pair of items x_1, $x_2 \in M$ the following is true: if $x_1 < x_2$, then $f(x_1) < f(x_2)$.

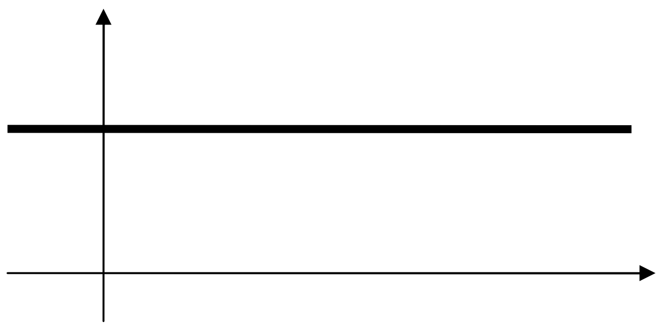


Function *f* is a ***decreasing function*** on set M if and only if for each pair of items x_1, $x_2 \in M$ the following is true: if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Function *f* is a ***non- increasing function*** on set M if and only if for each pair of items x_1, $x_2 \in M$ the following is true: if $x_1 < x_2$, then $f(x_1) \geq f(x_2)$.

Function *f* is a ***non- decreasing function*** on set M if and only if for each pair of items x_1, $x_2 \in M$ the following is true: if $x_1 < x_2$, then $f(x_1) \leq f(x_2)$.

Function *f* is a ***constant function*** on set M if and only if for each pair of items x_1, $x_2 \in M$ the following is true: if $x_1 < x_2$, then $f(x_1) = f(x_2)$.



The function is ***purely monotonic*** when it is either increasing or decreasing. When it is non-increasing, non-decreasing or constant we call it ***monotonic.***

Cardinality

Suppose you want to define what it means for two sets to "have the same number of elements". One way to do this is to say that two sets "have the same number of elements" if and only if all the elements of one set can be paired with the elements of the other, in such a way that each element is paired with exactly one element. Accordingly, we can define two sets to "have the same number of elements" if there is a bijection between them. We say that the two sets have the same cardinality.

Inverse function

If f is a function from X to Y then an **inverse function** for f, denoted by f^{-1}, is a function in the opposite direction, from Y to X, with the property that a round trip (a composition) returns each element to itself. Not every function has an inverse; those that do are called ***invertible***.

As a simple example, if f converts a temperature in degrees Celsius to degrees Fahrenheit, the function converting degrees Fahrenheit to degrees Celsius would be a suitable f^{-1}.

Composition of functions

One idea of enormous importance in all of mathematics is composition of functions: if *z* is a function of *y* and *y* is a function of *x*, then *z* is a function of *x*.

Note: z(y(x))

We may describe it informally by saying that the composite function is obtained by using the output of the first function as the input of the second one. This feature of functions distinguishes them from other mathematical constructs, such as numbers or figures, and provides the theory of functions with its most powerful structure.

The **function composition** of two or more functions uses the output of one function as the input of another. For example, $f(x) = sin(x^2)$ is the composition of the sine function and the squaring function. The functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ can be *composed* by first applying f to an argument x to obtain $y = f(x)$ and then applying g to y to obtain $z = g(y)$. The composite function formed in this way from general f and g may be written f(g(x)).

The function on the right acts first and the function on the left acts second, reversing English (and Slovak as well) reading order. We remember the order by reading the notation as "g of f". The order is important, because rarely do we get the same result both ways. For example, suppose $f(x) = x^2$ and $g(x) = x+1$. Then $g(f(x)) = x^2+1$, while $f(g(x)) = (x+1)^2$, which is x^2+2x+1, a different function.

The notation for composition reminds us of multiplication; in fact, sometimes we denote it using juxtaposition, gf, without an intervening circle. Under this analogy, identity functions are like 1, and inverse functions are like reciprocals (hence the notation).

The composition of two injections is again an injection, but if $g \circ f$ is injective, then it can only be concluded that f is injective. Every embedding is injective.

The composition of two surjections is again a surjection, but if $g \circ f$ is surjective, then it can only be concluded that g is surjective.

Bijective composition: the first function need not be surjective and the second function need not be injective.

The composition of two bijections is again a bijection, but if $g \circ f$ is a bijection, then it can only be concluded that f is injective and g is surjective. (See the remarks above regarding injections and surjections.)

The bijections from a set to itself form a group under composition, called the *symmetric group.*