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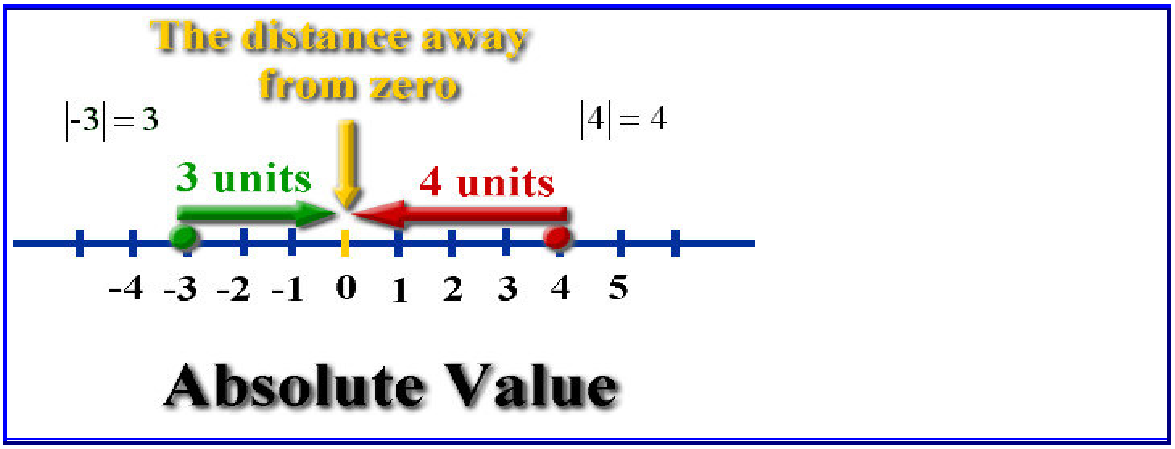
LINEAR FUNCTIONS WITH ABSOLUTE VALUE

The **absolute value** of a number can be considered as the distance between 0 and that number on the real number line.



Remember that distance is always a positive quantity (or zero). The distance in the diagram above from +4 to 0 is 4 units and the distance from -3 to 0 is 3 units. These units are never negative values. Using absolute value, we write this as: $|4| = 4$ and $|-3| = 3$

Proposition 1: $|a| = \sqrt{a^2}$

The absolute value has the following four fundamental properties:

Proposition 2:

$\|a\| \geq 0$	Non-negativity
$\|a\| = 0 \iff a = 0$	Positive-definiteness
$\|ab\| = \|a\|\|b\|$	Multiplicativeness
$\|a+b\| \leq \|a\| + \|b\|$	Subadditivity

Other important properties of the absolute value include:

Proposition 3:

$\lvert -a\rvert = \lvert a\rvert$	Symmetry
$\lvert a-b\rvert = 0 \iff a = b$	Identity of indiscernibles (equivalent to positive-definiteness)
$\lvert a-b\rvert \le \lvert a-c\rvert + \lvert c-b\rvert$	Triangle inequality (equivalent to subadditivity)
$\lvert a/b\rvert = \lvert a\rvert/\lvert b\rvert \text{ (if } b \ne 0)$	Preservation of division (equivalent to multiplicativeness)
$\lvert a-b\rvert \ge \lvert\lvert a\rvert - \lvert b\rvert\rvert$	(equivalent to subadditivity)

Two other useful inequalities are:

$$\lvert a\rvert \le b \iff -b \le a \le b$$
$$\lvert a\rvert \ge b \iff a \le -b \text{ or } b \le a$$

The above are often used in solving inequalities; for example:

$$\begin{aligned}\lvert x-3\rvert \le 9 &\iff -9 \le x-3 \le 9\\ &\iff -6 \le x \le 12\end{aligned}$$

EQUATIONS WITH THE ABSOLUTE VALUE

For each real number a it is true that:

if $a \geq 0$ then $\lvert a\rvert = a$

if $a < 0$ then $\lvert a\rvert = -a$

For each real number a is $\lvert a\rvert$ a nonnegative number. Therefore if a is a negative number, -a is a positive number.

Equations (inequalities) in which the unknown is in the absolute value are solved through dividing the domain into such intervals, in which each of the expression does not change the sign. In such integrals we solve that equation which is equivalent to the original one and does not contain the absolute value. Finally the solutions have to be compared and discussed with the definition intervals.

Graphing Absolute-Value Functions

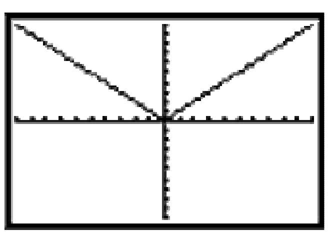
Taking the absolute value of a negative number makes it positive. For this reason, graphs of absolute values tend not to look quite as you're used to graphs looking, and it is important to include negative inputs in your chart. If you do not pick *x*-values that will put negatives inside the absolute value, you will mislead yourself as to what the graph looks like.

Examples:

To see what the graph of $y = |x|$ looks like let's create a table of values:

x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

To graph these values, simply plot the points and see what happens.



Whenever you have an **absolute value** graph, the general shape will look like a "v" (or in some cases, an upside down "v" as we will see later).

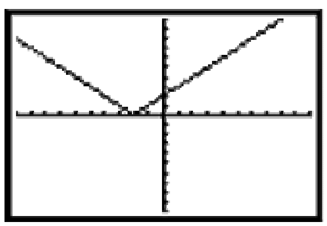
Let's Practice:

1. Graph y = |x+2|

   We know what the general shape should look like, but let's create a table of values to see exactly how this graph will look.

x	y
-3	|-3 + 2| = |-1| = 1
-2	|-2 + 2| = |0| = 0
-1	|-1 + 2| = |1| = 1
0	|0 + 2| = |2| = 2
1	|1 + 2| = |3| = 3
2	|2 + 2| = |4| = 4
3	|3 + 2| = |5| = 5

So our graph of y = |x + 2| looks like



Notice that the graph in this example looks almost identical to the graph of y = |x| except that it **was shifted to the left 2 units**.

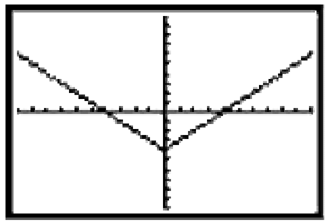
2. Graph y = |x| - 4

The table of values looks like this:

x	y
-5	5 - 4 = 1
-4	4 - 4 = 0
-3	3 - 4 = -1
-2	2 - 4 = -2

0	0 - 4 = -4
1	1 - 4 = -3

Which makes the graph look like this:



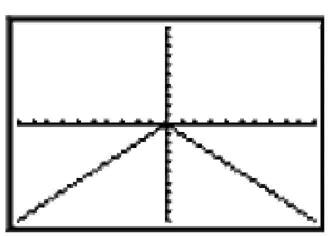
Notice that the graph in this example is the same shape as except that it has been **moved down 4 units**.

3. Graph y = -|x|

   In creating the table of values, be careful of your order of operations. You should find the absolute value of x first and then change the sign of that answer.

x	|x|	y
-3	3	-3
-2	2	-2
-1	1	-1
0	0	0
1	1	-1
2	2	-2
3	3	-3

So the graph looks like:



In this example, we have the exact same shape as the graph of y = |x| only the "v" shape is upside down now.

Based on the examples we've seen so far, there appears to be a pattern when it comes to graphing absolute value functions.

- When you have a function in the form **y = |x + h|** the graph will move h units to the left.
  When you have a function in the form **y = |x - h|** the graph will move h units to the right.

- When you have a function in the form **y = |x| + k** the graph will move up k units.
  When you have a function in the form **y = |x| - k** the graph will move down k units.

- If you have a negative sign in front of the absolute value, the graph will be reflected, or flipped, over the x-axis.

Keep in mind that you can also have combinations that change the absolute value graph more than once.

Solve the following examples:

Graph each of the following functions. You should try to use the rules shown above, but if you want to check yourself, make a table of values to make sure you are on the right track.

y = |x - 3|

y = |x + 1| - 3

y = -|x| + 2

y = |x - 2| + 1

y = -|x + 4|

y = |x - 1| + 3

SOLVED EXAMPLES:

EX 1 Create a graph of function y = $|3-x|$, $x \in R$

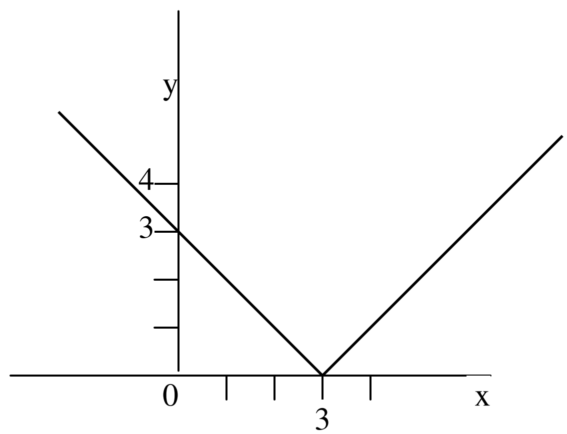
First we have to find out for which $x \in R$ it's valid: $|3-x|$ = 3 - x and for which $x \in R$ it's valid $|3-x|$ = x – 3.

Then we can write function as sum of two functions:

$3 - x \geq 0$ if $x \leq 3$ it means y = 3 – x for $x \in (-\infty, 3\rangle$

$3 - x \leq 0$ if $x \geq 3$ it means y = x – 3 for $x \in \langle 3, \infty)$

Graph of function:



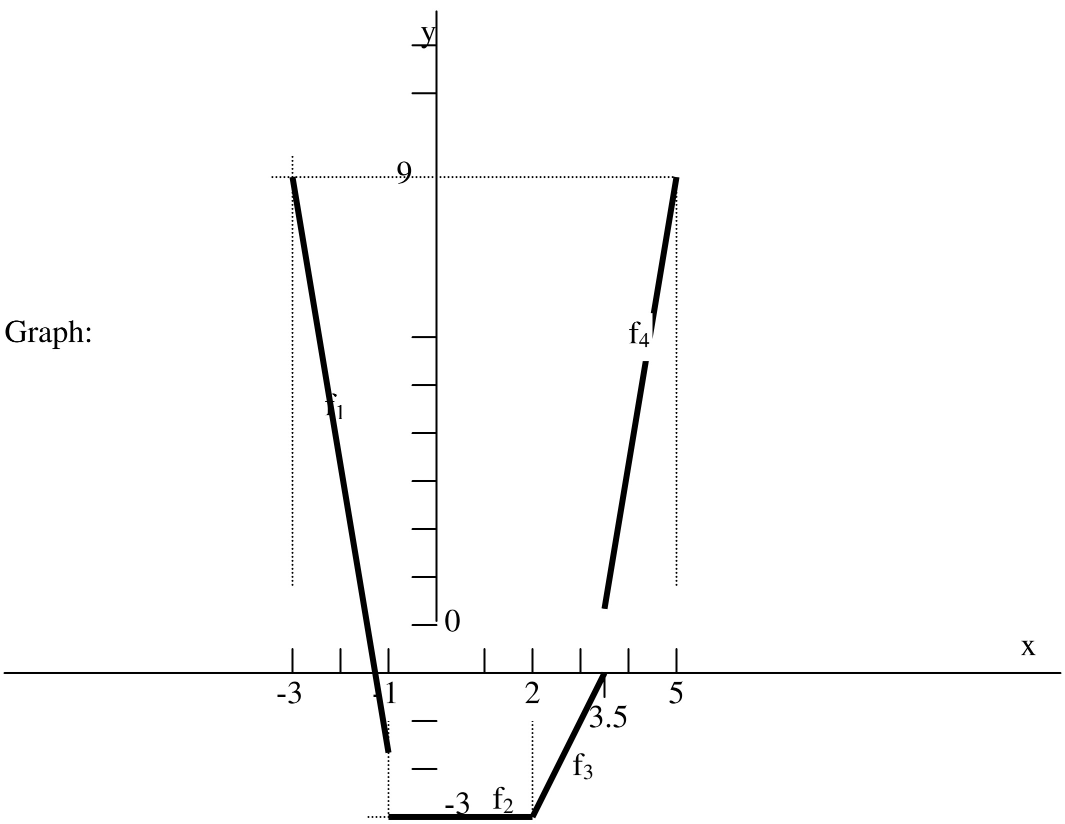
EX 2 Create a graph of function y = $|2-x| + |2x-7| + 3|1+x| - 15$ $\quad x \in \langle -3,5\rangle$

First we have to define all points at which the terms in absolute value equal 0. These are {2; 3.5; -1}. So the domain $\langle -3,5\rangle$ is divided into four intervals $\langle -3,-1\rangle$; $\langle -1,2\rangle$; $\langle 2,3.5\rangle$; $\langle 3.5,5\rangle$.

Then we can express absolute values of terms on each of four intervals and next functions f_1, f_2, f_3, f_4 for terms without absolute values.

Interval	$\langle -3,-1\rangle$	$\langle -1,2\rangle$	$\langle 2,3.5\rangle$	$\langle 3.5,5\rangle$
Term $\lvert 2-x\rvert$	2 – x	2 – x	x – 2	x – 2
Term $\lvert 2x-7\rvert$	7 – 2x	7 – 2x	7 – 2x	2x – 7
Term $\lvert 1+x\rvert$	-1 – x	1 + x	1 + x	1 + x
Term $3\lvert 1+x\rvert$	-3 – 3x	3 + 3x	3 + 3x	3 + 3x
$\lvert 2-x\rvert+\lvert 2x-7\rvert+3\lvert 1+x\rvert$ - 15	- 6x – 9	- 3	2x – 7	6x – 21
Function	f_1: y = -6x – 9 x∈ $\langle -3,-1\rangle$	f_2: y = - 3 x∈ $\langle -1,2\rangle$	f_3: y = 2x – 7 x∈ $\langle 2,3.5\rangle$	f_4: y = 6x – 21 x∈ $\langle 3.5,5\rangle$

Graph:



QUADRATIC FUNCTIONS WITH ABSOLUTE VALUE

To draw the graphs of quadratic functions with absolute value, we will follow the same steps as with the linear functions with absolute value, i.e.

1. set the zero points
2. make the table - Find the signs of expressions from the absolute values on individual intervals
3. calculate the function on individual intervals
4. draw the graph

Example 1: f: $y = |x^2 - 1|$

1. zero points: $x^2 - 1 = 0$

   $x^2 = 1$

   $|x| = 1$

   $x = \pm 1$

2. table:

	$(-\infty; -1)$	$\langle -1; 1)$	$\langle 1; \infty)$
$x^2 - 1$	-	+	-
	I.	II.	III

3. I. + III. $y = -(x^2 - 1)$

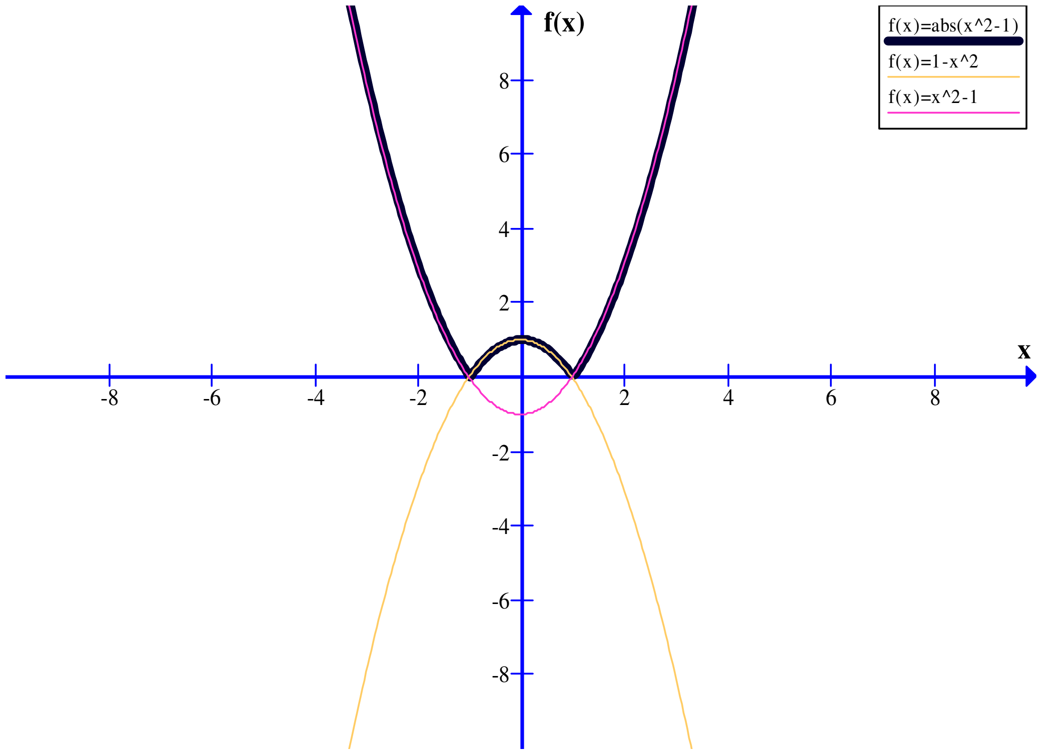
   $y = 1 - x^2$

   $\Leftrightarrow x \in (-\infty; -1) \cup \langle 1; \infty)$

   II. $y = x^2 - 1$

   $\Leftrightarrow x \in \langle -1; 1)$

4. graph:



Example 2: $f: y = |x^2 + 4x|$