Determinants, unlike matrices, **always** consist of a **square array** of elements.

The determinant of the square matrix **A** is denoted either by $|\mathbf{A}|$ or by det **A**.

Because determinants are always square, the expansion method just described can be applied to determinants of any size. Thus to evaluate the determinant of a 4×4 matrix, we first expand it along its top row to get an expression involving four 3×3 matrices, remembering to **alternate the plus and minus signs**. For example,

$$\begin{vmatrix} 1 & 3 & 4 & 2 \\ 5 & -1 & -3 & -4 \\ 2 & -3 & 4 & 7 \\ 1 & 8 & 5 & 6 \end{vmatrix} = 1\begin{vmatrix} -1 & -3 & -4 \\ -3 & 4 & 7 \\ 8 & 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 5 & -3 & -4 \\ 2 & 4 & 7 \\ 1 & 5 & 6 \end{vmatrix} + 4\begin{vmatrix} 5 & -1 & -4 \\ 2 & -3 & 7 \\ 1 & 8 & 6 \end{vmatrix} - 2\begin{vmatrix} 5 & -1 & -3 \\ 2 & -3 & 4 \\ 1 & 8 & 5 \end{vmatrix}$$

We then proceed to evaluate each 3×3 matrix as before.

Rules for the manipulation of determinants

Changing a determinant without changing its value

We can alter the rows and the columns of a determinant in three ways **without changing its value**. Two are given below.

Adding any row, or column, to any other row, or column

If we add the corresponding elements in two rows (or columns), the value of the determinant is unaltered. For example, we have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+b & b & c \\ d+e & e & f \\ g+h & h & i \end{vmatrix}$$

The rule also applies to the **subtraction** of the corresponding elements in two rows (or columns). So, we have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d-g & e-h & f-i \\ g & h & i \end{vmatrix}$$

Example 2 Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 4 & 6 & 8 \end{vmatrix}$

SOLUTION

The most efficient way to evaluate this determinant is to add the second row to the first row.

Note If you cannot quickly spot this simplification, it is better to expand using 2×2 determinants, rather than to spend time trying various possible simplifications.

So, we have

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 4 & 6 & 8 \end{vmatrix} = \begin{vmatrix} 1+0 & 1-1 & 1-1 \\ 0 & -1 & -1 \\ 4 & 6 & 8 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 4 & 6 & 8 \end{vmatrix}$$

Expanding this simplified determinant, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 4 & 6 & 8 \end{vmatrix} = 1 \times \begin{vmatrix} -1 & -1 \\ 6 & 8 \end{vmatrix} = 1 \times (-8+6) = -2$$

Adding any multiple of any row, or column, to any other row, or column

If we add the same multiple of the elements of a column (or row) to the corresponding elements of another column (or row), the value of the determinant is unaltered. For example, we have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+5b & b & c \\ d+5e & e & f \\ g+5h & h & i \end{vmatrix}$$

The rule also applies to negative multiples. So, we have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d-3a & e-3b & f-3c \\ g & h & i \end{vmatrix}$$

Example 3 Evaluate $\begin{vmatrix} 4 & 6 & 8 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{vmatrix}$

SOLUTION

This determinant is best simplified by subtracting $2 \times$ the third row from the first row.

Again, if you cannot quickly spot this, it is better to expand using 2×2 determinants.

So, we have

$$\begin{vmatrix} 4 & 6 & 8 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 4-2 & 6-6 & 8-8 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{vmatrix}$$

Expanding this simplified determinant, we get

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{vmatrix} = 2 \times \begin{vmatrix} 1 & 4 \\ 3 & 4 \end{vmatrix} = -16$$

We can take the factor k out of each column. Hence, we obtain

$$\begin{vmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{vmatrix} = k^3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Transpose of a determinant

The **transpose** of a determinant is obtained by reflecting the determinant in its **leading diagonal**. (This is the diagonal from the top left corner to the bottom right corner. It is also known as the **principal diagonal**.)

The value of the transpose of a determinant is the **same** as the determinant's **original value**. For example, we have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

Example 5 Evaluate $\begin{vmatrix} 2 & 8 & 9 \\ 0 & -1 & 3 \\ 0 & 4 & 1 \end{vmatrix}$

SOLUTION

To simplify the calculation, we replace the given determinant by its transpose:

$$\begin{vmatrix} 2 & 8 & 9 \\ 0 & -1 & 3 \\ 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 8 & -1 & 4 \\ 9 & 3 & 1 \end{vmatrix}$$

which gives

$$\begin{vmatrix} 2 & 0 & 0 \\ 8 & -1 & 4 \\ 9 & 3 & 1 \end{vmatrix} = 2\begin{vmatrix} -1 & 4 \\ 3 & 1 \end{vmatrix} = 2(-1 - 12) = -26$$

Factorisation of determinants

The easier way to find the factors of a determinant is to use the rules for manipulating determinants. Rarely, if ever, do we multiply out the determinant and then factorise the result.

In Example 6, the factors are obtained by subtracting, in turn, one column from another. In Example 7, a factor is obtained by first adding **all** three rows.

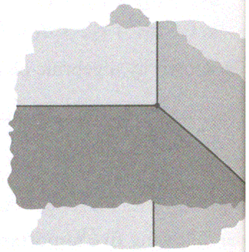
Example 6 Factorise $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

Geometric interpretation of three equations in three unknowns

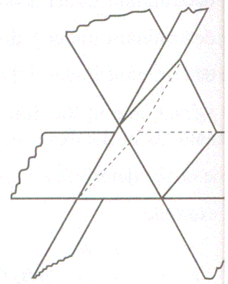
Each of the equations $a_ix + b_iy + c_iz + d_i = 0$ $(i = 1, 2, 3)$ may be considered as the equation of a plane in three-dimensional space.

With three planes, there are seven possible configurations.

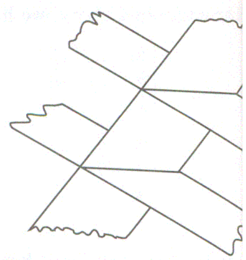
- The three planes intersect in a single point. In this case, the three equations have a **unique solution**.



- The three planes form a triangular prism. In this case, there is no point where all three planes intersect. Hence, the equations are said to be **inconsistent**, as they have no solutions.



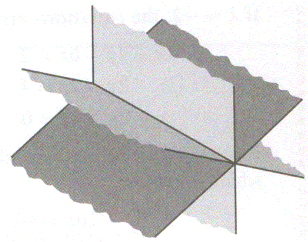
- Two of the planes are parallel and separate, and are intersected by the third plane. Again, there is no point where all three planes intersect, and so the equations are inconsistent in this case, too.



- Two other configurations in which the planes have no common point and therefore their equations are inconsistent are:
  - ○ All three planes are parallel and separate.
  - ○ Two of the planes are coincident and the third plane is parallel but separate.

  The two remaining configurations correspond to the three equations having infinitely many solutions.

- The three planes have a common line, giving an infinite number of points (x, y, z) which satisfy all three equations. In this case, the equations are said to be **linearly dependent**, and the configuration is called a **sheaf of planes** or a **pencil of planes**.



- All three planes coincide, giving an infinite number of points which satisfy all three equations.

Example 8 How many solutions are there to these three equations?

$$4x - \lambda y + 6z = 2$$
$$2y + \lambda z = 1$$
$$x - 2y + 4z = 0$$

SOLUTION

First, we find the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 4 & -\lambda & 6 \\ 0 & 2 & \lambda \\ 1 & -2 & 4 \end{vmatrix}$$

$$= 4\begin{vmatrix} 2 & \lambda \\ -2 & 4 \end{vmatrix} + \lambda\begin{vmatrix} 0 & \lambda \\ 1 & 4 \end{vmatrix} + 6\begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= 4(8 + 2\lambda) + \lambda(-\lambda) + 6(-2)$$

$$= -\lambda^2 + 8\lambda + 20$$

Therefore, there is a unique solution unless

$$-\lambda^2 + 8\lambda + 20 = 0$$
$$\Rightarrow \quad \lambda^2 - 8\lambda - 20 = 0$$
$$\Rightarrow \quad (\lambda - 10)(\lambda + 2) = 0$$

That is, there is a unique solution unless $\lambda = 10$ or $\lambda = -2$.

If $\lambda = 10$, the equations are

$$4x - 10y + 6z = 2 \qquad [1]$$
$$2y + 10z = 1 \qquad [2]$$
$$x - 2y + 4z = 0 \qquad [3]$$

We now use equations [1] and [3] to get the same expression as that on the left-hand side of equation [2]. Subtracting $4 \times$ equation [3] from equation [1], we have

$$-2y - 10z = 2$$
$$\Rightarrow \quad 2y + 10z = -2$$

This contradicts equation [2], and so the equations have no solution. That is, the three equations are **inconsistent**.

If $\lambda = -2$, the equations are

$$4x + 2y + 6z = 2 \qquad [4]$$
$$2y - 2z = 1 \qquad [5]$$
$$x - 2y + 4z = 0 \qquad [6]$$

Proceeding as before, we subtract equation [4] from $4 \times$ equation [6], which gives

$$-10y + 10z = -2$$
$$\Rightarrow \quad 2y - 2z = \frac{1}{5}$$

This contradicts equation [5], and so the equations have no solution. That is, the three equations are inconsistent.

Example 9 Solve the equations

$$2x - 3y + 4z = 1 \qquad [1]$$
$$3x - y = 2 \qquad [2]$$
$$x + 2y - 4z = 1 \qquad [3]$$

SOLUTION

First, we calculate the determinant $\begin{vmatrix} 2 & -3 & 4 \\ 3 & -1 & 0 \\ 1 & 2 & -4 \end{vmatrix}$ and find that its value is zero.

Therefore, there is not a unique solution to the three equations, and so we cannot use the general formula for the solution of three equations.

Adding equations [1] and [3], we obtain

$$3x - y = 2$$

which is equation [2].

Since one equation is a combination of the other two, the equations are said to be **linearly dependent**.

We cannot find a unique solution for two equations in three unknowns.

To solve these equations, we let $x = t$. Hence, x is no longer an unknown. We thereby have only two unknowns in these two equations, and so we can solve them.

Using equation [2], we obtain $y = 3t - 2$. Substituting this in equation [3], we get

$$4z = t + 2(3t - 2) - 1$$
$$\Rightarrow \quad z = \frac{7t - 5}{4}$$

So, the solution is $\left(t, 3t - 2, \dfrac{7t - 5}{4}\right)$.

Each value of the parameter t gives a different point. Since there is only one parameter, this solution represents a line.

A zero matrix may have **any** order and therefore is not unique. For example,

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We can multiply any non-zero matrix by a zero matrix provided the zero matrix is **conformable for multiplication**. For example,

$$\begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and $$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Generally, we have

$$\mathbf{0M} = \mathbf{0} \quad \text{and} \quad \mathbf{N0} = \mathbf{0}$$

Also, from the second example, we note that when $\mathbf{0}$ and $\mathbf{M}$ have the **same order**

$$\mathbf{0M} = \mathbf{0} = \mathbf{M0}$$

which is one of the three exceptions to the non-commutative laws discussed on page 302.

When we multiply together two **non-zero matrices**, we can get a **zero matrix** as the result. For example,

$$\begin{pmatrix} 5 & 2 \\ 10 & 4 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -5 & -10 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Inverse matrices

If $\mathbf{M}$ is a square matrix, its **inverse**, denoted by $\mathbf{M}^{-1}$, is defined by

$$\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

Contrary to the non-commutative law discussed on page 302, we note that the order in which we multiply $\mathbf{M}$ and $\mathbf{M}^{-1}$ does not matter, which means that $\mathbf{M}^{-1}$, if it exists, is unique.

The inverse of a square matrix, $\mathbf{M}$, exists when $\det \mathbf{M} \neq 0$. That is, when $\mathbf{M}$ is said to be **non-singular**. When $\det \mathbf{M} = 0$, $\mathbf{M}$ is said to be **singular**.

The minor determinant

The **minor determinant** of an element of a matrix is the determinant of the matrix formed by deleting the row and column containing that element.

For example, the minor determinant of the middle element, 2, of the matrix $\begin{pmatrix} 5 & 6 & 9 \\ 7 & 2 & 1 \\ 3 & 4 & 8 \end{pmatrix}$ is the determinant of the matrix $\begin{pmatrix} 5 & 9 \\ 3 & 8 \end{pmatrix}$, which is

$$\begin{vmatrix} 5 & 9 \\ 3 & 8 \end{vmatrix} = 13$$