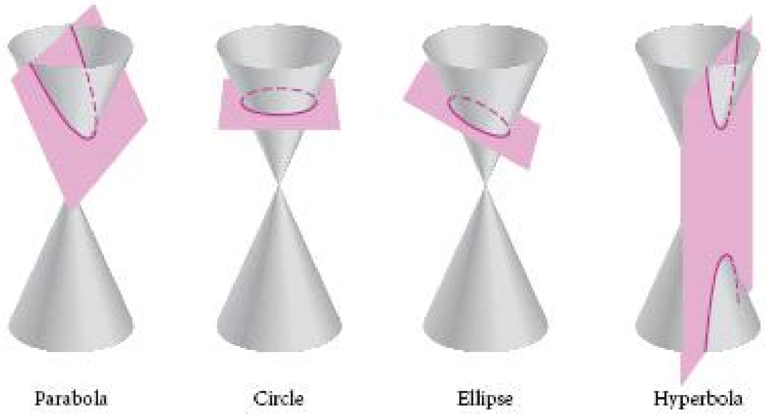
Conic sections (Conics)

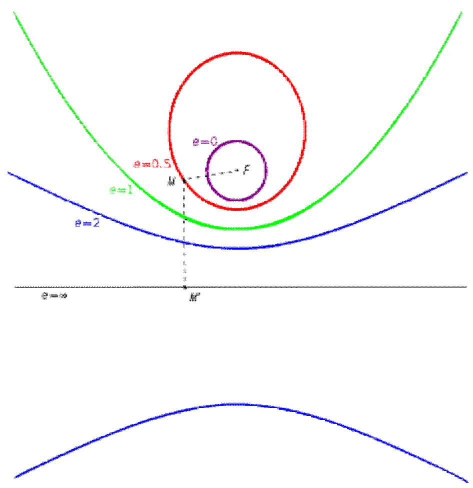
A **conic section** is formed when a right circular cone with two parts, called *nappes*, is intersected by a plane. One of four types of curves can be formed: a parabola, a circle, an ellipse, or a hyperbola.
A circle is formed when a cone is cut perpendicular to its axis. **An ellipse** is produced when the cone is cut obliquely to the axis and the surface. **A hyperbola** results when the cone is intersected by a plane parallel to the axis, and **a parabola** results when the intersecting plane is parallel to an element of the surface.



Conic Sections

When such a curve is plotted on a coordinate system, it may be defined as follows:

Definition
A conic section is the locus of all points in a plane whose distance from a fixed point is a constant ratio to its distance from a fixed line. The fixed point is the focus, and the fixed line is the directrix.

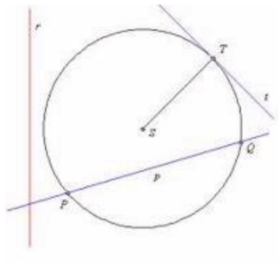


The ratio referred to in the definition is called **the eccentricity (e).**

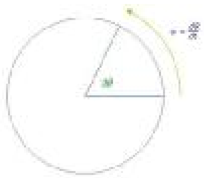
$0 < e < 1$, it is an ellipse
$e > 1$, it is a hyperbola
$e = 1$, it is a parabola
$e = 0$, it is a circle.

Circle

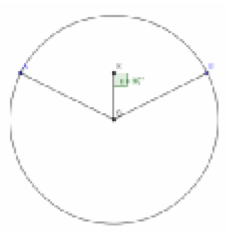
Basic terminology:



A **chord** is an abscissa that joins two points on the circumference of a circle.
A **tangent** to a circle is a straight line that touches the circle at one point (the point of tangency).
An **exterior line** to a circle is a straight line that neither touches nor cuts the circle.
A **diameter** is a chord that passes through the centre.
Circumference is the distance measured around the curved edge of a circle.
A **radius** is any abscissa from the centre to the circumference.

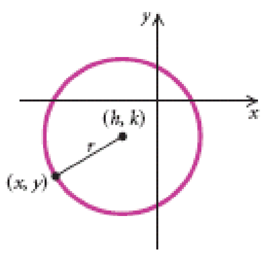


The curved part of the circle between points A and B is known as an **arc**.
There are two arcs AB: The larger is the **major arc**, the smaller is the **minor arc**.
The chord AB divides the circle into two **segments**, the major segment and the minor segment.



The **perpendicular bisector** of any chord passes through the centre of its circle (i.e. it is a line that is perpendicular to the chord and passes through its centre).
Any straight line that cuts across a circle at two points is called a **secant**. (A tangent is a special case of a secant, as the two cuts become a single point at contact. These two cuts are said to become coincident.)

> **Definition:**
> **The locus of all points in the plane, the distance of whose from a fixed point S, called the *centre*, is a constant, called a *radius*. SX = r = X – S**



Equations of a circle:
The equation of the circle is centred at the origin with radius r is given by: $\mathbf{x^2 + y^2 = r^2}$
If X [x, y] is a general point on the circumference and the centre S [h, k], then the analytical expression of a circle with the centre S and radius r is:

$$\mathbf{(x - h)^2 + (y - k)^2 = r^2}$$

This equation is called the *centre form of the circle equation*.
When expanded the brackets and simplified, we get *the general form of the circle equation:*

$$\mathbf{x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0}, \text{ what is in fact: } \mathbf{x^2 + y^2 + ax + by + c = 0.}$$

If point X lies:

a) On the circle, then $\mathbf{(x - m)^2 + (y - n)^2 = r^2}$

b) Inside the circle, then $\mathbf{(x - m)^2 + (y - n)^2 < r^2}$

c) Outside the circle, then $\mathbf{(x - m)^2 + (y - n)^2 > r^2}$

Example1:

Find the equation of the circle with centre $[8,-7]$ and radius 9.

The equation is $\mathbf{(x - 8)^2 + (y + 7)^2 = 9^2}$.

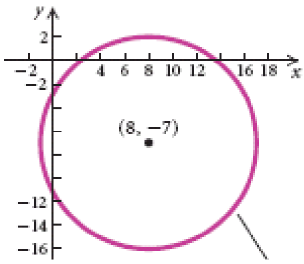
Expanding and simplifying give the general form:

$x^2 - 16x + 64 + y^2 + 14y + 49 = 81$

$x^2 - 16x + y^2 + 14y + 113 = 81$ /-81

$x^2 - 16x + y^2 + 14y + 32 = 0$

$x^2 + y^2 - 16x + 14y + 32 = 0$



If the equation of a circle is given in the general form then we can use the technique of completing the square to express it in the centre form of the circle equation.

Example2:

Find the centre and radius of the circle with equation $x^2 + y^2 + 2x - 4y - 4 = 0$.

Group x terms together and y terms together:

$x^2 + 2x + y^2 - 4y - 4 = 0$

Complete the square for the x terms and the y terms:

$(x+1)^2 - 1 + (y-2)^2 - 4 - 4 = 0$

$(x+1)^2 + (y-2)^2 - 9 = 0$ /+9

$(x+1)^2 + (y-2)^2 = 9$

The centre is $[-1,2]$ and the radius is $\sqrt{9} = 3$.

Example3:

Find the equation of the circle whose centre is at the point $[1,3]$ and which passes through the point $[4,3]$.

The centre is at $[1,3]$, so: $(x-1)^2 + (y-3)^2 = r^2$

For the point $[4,3]$, so: $(4-1)^2 + (3-3)^2 = r^2$

$9 = r^2$, so $r = 3$

The equation of the circle is: $(x-1)^2 + (y-3)^2 = 9$

Example 4:

Find the equation of the circle which passes through the points A $[1,2]$, B $[2,5]$ and C $[-3,4]$.

The circle has equation: $\mathbf{x^2 + y^2 + ax + by + c = 0}$

For the point $[1,2]$, so: $1^2 + 2^2 + 1a + 2b + c = 0 \quad [1]$

For the point $[2,5]$, so: $2^2 + 5^2 + 2a + 5b + c = 0 \quad [2]$

For the point $[-3,4]$, so: $(-3)^2 + 4^2 - 3a + 4b + c = 0 \quad [3]$

Solve simultaneously

$$5 + 1a + 2b + c = 0 \quad [1]$$
$$29 + 2a + 5b + c = 0 \quad [2]$$
$$25 - 3a + 4b + c = 0 \quad [3]$$

Express from $[1]$ $a = -2b - c - 5$ and put it to $[2]$, $[3]$.

$$29 + 2a + 5b + c = 0$$
$$25 - 3a + 4b + c = 0$$

$$29 + 2(-2b - c - 5) + 5b + c = 0$$
$$25 - 3(-2b - c - 5) + 4b + c = 0$$

$$29 - 4b - 2c - 10 + 5b + c = 0$$
$$25 + 6b + 3c + 15 + 4b + c = 0$$

$$b - c + 19 = 0 \quad /(-10)$$
$$10b + 4c + 40 = 0$$

$$-10b + 10c - 190 = 0$$
$$10b + 4c + 40 = 0$$

$$14c = 150$$

$$c = \frac{150}{14}$$

$$c = \frac{75}{7}$$

$$c = \frac{75}{7}$$

$$b - c + 19 = 0$$

$$b - \left(\frac{75}{7}\right) + 19 = 0$$

$$b + \frac{-75 + 133}{7} = 0$$

$$b = -\frac{58}{7}$$

Substituting into $[1]$ $a = -2b - c - 5$

$$a = -2\left(\frac{-58}{7}\right) - \frac{75}{7} - 5 = \frac{116 - 75 - 35}{7} = \frac{6}{7}$$

The equation of the circle is: $x^2 + y^2 + \frac{6}{7}x - \frac{58}{7}y + \frac{75}{7} = 0 \quad / 7$

$$7x^2 + 7y^2 + 6x - 58y + 75 = 0$$

Exercise

1. Find the equation of the circles with these centres and radii.

a) Centre [1,2], radius 3
b) Centre [3,1], radius 4
c) Centre [−2,3], radius 1
d) Centre [1,−3], radius 5
e) Centre [−4,0], radius 4
f) Centre [2,−4], radius 7
g) Centre [−3,5], radius 6
h) Centre [4,−1], radius 3

2. Find the centre and radius of each of these circles.

a) $x^2 + y^2 = 16$
b) $x^2 + y^2 = 81$
c) $x^2 + y^2 + 6x - 4y + 12 = 0$
d) $x^2 + y^2 - 4x = 0$
e) $x^2 + y^2 + 6y - 16 = 0$
f) $x^2 + y^2 - 6x + 8y - 11 = 0$
g) $x^2 + y^2 + 14x - 10y - 7 = 0$
h) $x^2 + y^2 - 12x - 12y + 8 = 0$
i) $x^2 + y^2 + 16x + 12y = 0$
j) $x^2 + y^2 - 2x + 2y - 2 = 0$
k) $x^2 + y^2 - 14x + 16y - 31 = 0$
l) $x^2 + y^2 - 5y + 4 = 0$

3. Find the equation of the circle whose

a) centre is at point [5,4] and which passes through the point [9,7]
b) centre is at point [1,−7] and which passes through the point [−4,5]
c) centre is at point [5,7] and which touches the x-axis
d) centre is at point [−2,−3] and which touches the y-axis

4. Find the equation of the circle which has the points $A[2,5]$ and $B[10,11]$ as the ends of a diameter.

5. Find the equation of the circle which has the points $P[-2,3]$ and $Q[4,5]$ as the ends of a diameter.

6. Find the equations of the circles of radius 5, which touch the x-axis and pass through the point $B[3,1]$

7. Determine whether the points $A[1,2]$, $B[8,-2]$ and $C[7,10]$ lie on the circle whose equation is $x^2 + y^2 - 8x - 12y + 27 = 0$

8. Determine whether the points $A[0,6]$ and $B[4,-2]$ lie on the circle whose equation is $x^2 + y^2 - 2x + 2y = 8$

9. Determine whether the points $P[2,-1]$ and $Q[3,7]$ lie on the circle whose equation is $x^2 + y^2 - 12x - 4y + 15 = 0$

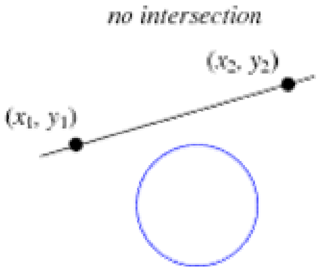
10. Find the equation of the circle whose centre which passes through the points:

a) A $[0,1]$, **B** $[4,3]$ **and C** $[1,-1]$.
b) A $[1,0]$, **B** $[0,1]$ **and C** $[3,4]$
c) A $[2,-1]$, **B** $[1,3]$ **and C** $[1,-4]$.
d) A $[2,2]$, **B** $[-2,1]$ **and C** $[2,3]$.

Intersection of a line and a circle

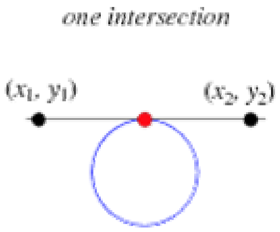
Consider a straight line $y = mx + c$ and a circle $(x-a)^2 + (y-b)^2 = r^2$.
To decide how many points of intersection there are we must solve simultaneously the equations of the straight line and the circle. The number of roots of the resulting quadratic equation determines which case applies:

1. No real roots



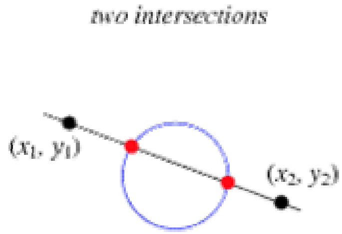
The straight line does not intersect the circle

2. One repeated root



The straight line touches the circle at one point – it is a tangent to the circle

3. Two distinct roots



The straight line intersects the circle at two distinct points

Example1:
Determine whether the line $y = x+1$ intersects the circle $x^2 + y^2 - 6x - 2y + 6 = 0$ at two points, one point or not at all.

Substitute $y = x+1$ in the equation of the circle:

$x^2 + (x+1)^2 - 6x - 2(x+1) + 6 = 0$

$x^2 + x^2 + 2x + 1 - 6x - 2x - 2 + 6 = 0$

$2x^2 - 6x + 5 = 0$

Using the discriminant with a = 2, b = -6, c = 5 gives

$D = b^2 - 4ac$

$D = (-6)^2 - 4(2)(5) = 36 - 40 = -4$

Since the discriminant is negative the quadratic equation has no real roots.
Geometrically this tells you that the line does not intersect the circle.

Example 2:
Determine whether the line $x + y = 3$ intersects the circle $x^2 + y^2 + x - 5y + 4 = 0$ at two points, one point or not at all. Find the coordinates of these points (the points of intersection or the point of tangency).

Substitute $y = -x+3$ in the equation of the circle:

$x^2 + (-x+3)^2 + x - 5(-x+3) + 4 = 0$

$x^2 + x^2 - 6x + 9 + x + 5x - 15 + 4 = 0$

$2x^2 - 2 = 0$

$x^2 - 1 = 0$

Using the discriminant with a = 1, b = 0, c = -1 gives

$$D = b^2 - 4ac$$
$$D = 0^2 - 4(1)(-1) = 0 + 4 = 4$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$$

$$x_{1,2} = \frac{0 \pm \sqrt{4}}{2} = \pm 1$$

This equation has two distinct roots and hence the straight line intersects the circle at two distinct points.

When **x = 1, y = -1 + 3 = 2** and **x = -1, y = 1 + 3 = 4**

The points of intersection are $[1,2]$ **and** $[-1,4]$**.**

Example 3:

Show that the line $y = x - 2$ **is a tangent to the circle** $x^2 + y^2 - 6x + 2y + 8 = 0$**. Find the coordinates of the point of tangency.**

Substitute $y = x - 2$ in the equation of the circle:

$$x^2 + (x-2)^2 - 6x + 2(x-2) + 8 = 0$$
$$x^2 + x^2 - 4x + 4 - 6x + 2x - 4 + 4 = 0$$
$$2x^2 - 8x + 8 = 0$$
$$x^2 - 4x + 4 = 0$$

Using the discriminant with a = 1, b = - 4, c = 4 gives

$$D = b^2 - 4ac$$
$$D = (-4)^2 - 4(1)(4) = 16 - 16 = 0$$

Since the discriminant is zero the quadratic equation has one repeated root.

The one repeated root implies that the line touches the circle at just one point. Therefore, it is a tangent to the circle.

The coordinates of the point of tangency

$$x = \frac{-b}{2a}$$

$$x = \frac{4}{2} = 2, \text{ y} = 2 - 2 = 0$$

The point of tangency is $[2,0]$**.**

Tangent to the circle

Definition

The analytical expression of a tangent to a circle **$(x – m)^2 + (y – n)^2 = r^2$** at **the point of tangency T** $[x_0, y_0]$ is $(x_0 - m)(x - m) + (y_0 - n)(y - n) = r^2$

Example 4:

Find the equation of the tangent to the circle $x^2 + y^2 - 6x - 4y + 3 = 0$ **at the point of tangency** $A[2,5]$**.**

Complete the square for the x terms and the y terms:

$(x-3)^2 - 9 + (y-2)^2 - 4 + 3 = 0$

$(x-3)^2 + (y-2)^2 - 10 = 0$ /+9

$(x-3)^2 + (y-2)^2 = 10$

for the centre $S[3,2]$ **and the radius r = 10:** $(x_0 - 3)(x - 3) + (y_0 - 2)(y - 2) = 10$

for the point of tangency $[2,5]$**:** $(2-3)(x-3) + (5-2)(y-2) = 10$

$$-1(x-3) + 3(y-2) = 10$$

$$-x + 3 + 3y - 6 = 10$$

$$-x + 3y - 13 = 0 \qquad \textbf{/ + x} \quad \textbf{/ + 13}$$

$$3y = x + 13 \qquad \textbf{/ : 3}$$

$$y = \frac{1}{3}x + \frac{13}{3}$$

The equation of the tangent to the circle at the point of tangency A is: $y = \frac{1}{3}x + \frac{13}{3}$**.**

Exercise

1. Determine whether the given straight line intersects the corresponding circle at two points, one point or not at all.

a) $y = x - 4$ and $x^2 + y^2 - 5x + 3y + 8 = 0$

b) $y = 3x - 11$ and $x^2 + y^2 - 4x - 30y + 209 = 0$

c) $y = 2x - 5$ and $x^2 + y^2 - 12x - 4y - 10 = 0$

d) $y = x + 2$ and $x^2 + y^2 + 3x - y = 0$

e) $y = x + 2$ and $x^2 + y^2 - 7x - 3y + 11 = 0$

f) $y = 2x - 1$ and $x^2 + y^2 - 8x + y + 25 = 0$

2. Determine whether the given straight line intersects the corresponding circle at two points, one point or not at all. Find the coordinates of these points (the points of intersection or the point of tangency).

a) $y = 7 - x$ and $x^2 + y^2 - 6x - 4y + 9 = 0$

b) $y = 7x + 2$ and $x^2 + y^2 + 8x + 2y - 8 = 0$

c) $y = 11 - x$ and $x^2 + y^2 - 8y - 9 = 0$

d) $y = 5x + 15$ and $x^2 + y^2 + 10x - 6y - 31 = 0$

3. Show that the line $y = x - 3$ **is a tangent to the circle** $x^2 + y^2 - 5x - 9y + 14 = 0$**. Find the coordinates of the point of tangency.**

4. Show that the line $y = 2x - 3$ **is a tangent to the circle** $(x-5)^2 + (y-2)^2 = 5$**. Find the coordinates of the point of tangency.**

5. Show that the line $y = 3x - 1$ **neither cuts nor touches the circle** $(x-4)^2 + (y-1)^2 = 9$

6. Show that the line $y = 3x + 1$ **is a tangent to the circle** $x^2 + y^2 - 14x - 4y + 13 = 0$**. Find the coordinates of the point of tangency.**

7. Show that the line $y = x + 3$ **neither cuts nor touches the circle** $x^2 + y^2 - 6x + 8y - 7 = 0$**.**

8. Show that the line $y = 2x + 3$ **is a secant to the circle** $(x-4)^2 + (y-1)^2 = 28$**.**

9. Show that the line $y = 3x + 5$ **is a secant to the circle** $x^2 + y^2 - 2x - 6y + 5 = 0$**.**

10. Find the equation of the tangent to the corresponding circle at the given point of tangency.

a) $x^2 + y^2 - 2x - 6y + 8 = 0$, **T** $[2,2]$

b) $x^2 + y^2 + 6x - 4y + 8 = 0$, **T** $[-1,1]$

c) $x^2 + y^2 + 10x + 8y + 39 = 0$, **T** $[-4,-3]$

d) $x^2 + y^2 + 10y + 20 = 0$, **T** $[2,-4]$

e) $x^2 + y^2 - 14x + 8y + 57 = 0$, **T** $[9,-2]$

f) $x^2 + y^2 - 12x - 16y = 0$, **T** $[0,0]$