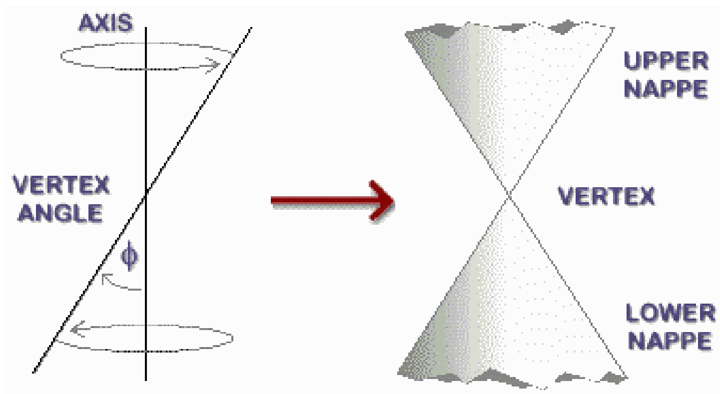
Conics

Conic sections are the curves which result from the intersection of a plane with a cone. These curves were studied and revered by the ancient Greeks, and were written about extensively by both Euclid and Appolonius. They remain important today, partly for their many and diverse applications.

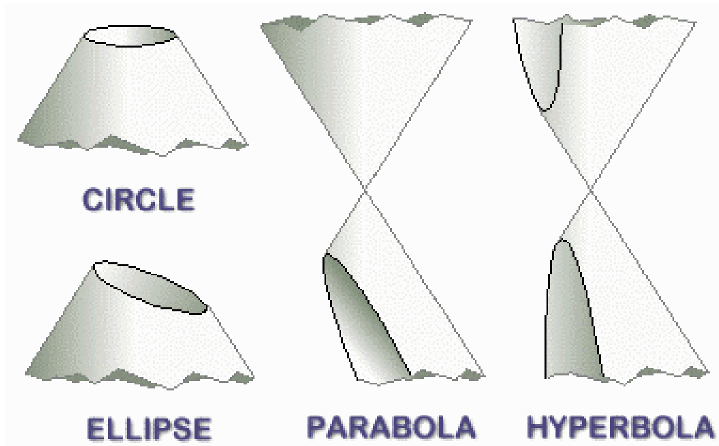
Although to most people the word "cone" conjures up an image of a solid figure with a round base and a pointed top, to a mathematician a cone is a *surface*, one which is obtained in a very precise way.

Imagine a vertical line, and a second line intersecting it at some angle φ (phi). We will call the vertical line the *axis*, and the second line the *generator*. The angle φ between them is called the *vertex angle*. Now imagine grasping the axis between thumb and forefinger on either side of its point of intersection with the generator, and twirling it. The generator will sweep out a surface, as shown in the diagram. It is this surface which we call a **cone**.



Notice that a cone has an upper half and a lower half (called the *nappes*), and that these are joined at a single point, called the *vertex*. Notice also that the nappes extend indefinitely far both upwards and downwards. A cone is thus completely determined by its vertex angle.

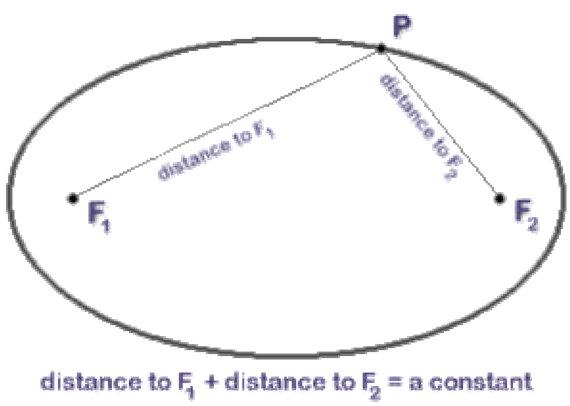
Now, in intersecting a flat plane with a cone, we have three choices, depending on the angle the plane makes to the vertical axis of the cone. First, we may choose our plane to have a greater angle to the vertical than does the generator of the cone, in which case the plane must cut right through one of the nappes. This results in a closed curve called an *ellipse*. Second, our plane may have exactly the same angle to the vertical axis as the generator of the cone, so that it is parallel to the side of the cone. The resulting open curve is called a *parabola*. Finally, the plane may have a smaller angle to the vertical axis (that is, the plane is steeper than the generator), in which case the plane will cut both nappes of the cone. The resulting curve is called a *hyperbola*, and has two disjoint "branches."



Notice that if the plane is actually perpendicular to the axis (that is, it is horizontal) then we get a circle – showing that a circle is really a special kind of ellipse. Also, if the intersecting plane passes through the vertex then we get the so-called *degenerate* conics; a single point in the case of an ellipse, a line in the case of a parabola, and two intersecting lines in the case of a hyperbola.

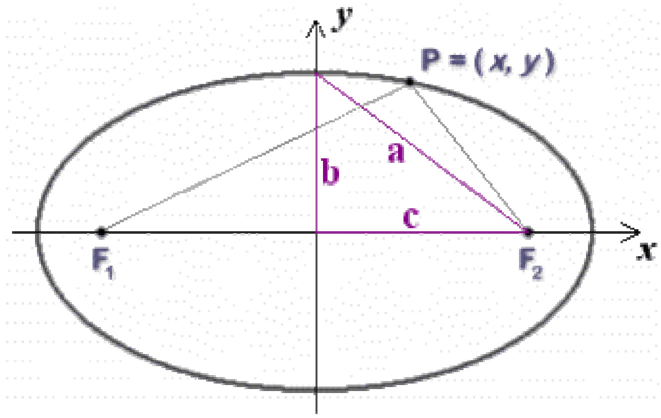
ELLIPSE

The set of all points in the plane, the *sum* of whose distances from two fixed points, called the *foci*, is a constant. ("Foci" is the plural of "focus)



For reasons that will become apparent, we will denote the sum of these distances by $2a$.

We see from the definition that an ellipse has two axes of symmetry, the larger of which we call the *major axis* and the smaller the *minor axis*. The two points at the ends of the ellipse (on the major axis) are called the *vertices*. It happens that the length of the major axis is $2a$, the sum of the distances from any point on the ellipse to its foci. If we call the length of the minor axis 2*b* and the distance between the foci 2c (where c = e, called ***eccentricity***), then the Pythagorean Theorem yields the relationship $b^2 + c^2 = a^2$:



By imposing coordinate axes in this convenient manner, we see that the vertices are at the *x* intercepts, at *a* and -*a*, and that the *y*-intercepts are at *b* and -*b*. Let the variable point *P* on the ellipse be given the coordinates (*x, y*). We may then apply the distance formula for the distances from *P* to F_1 and from *P* to F_2 to express our geometrical definition of the ellipse in the language of algebra:

$$\begin{aligned} 2a &= PF_1 + PF_2 \\ &= \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} \end{aligned}$$

Substituting $a^2 - b^2$ for c^2 and using a little algebra, we can then derive the standard equation for an ellipse centred at the origin,

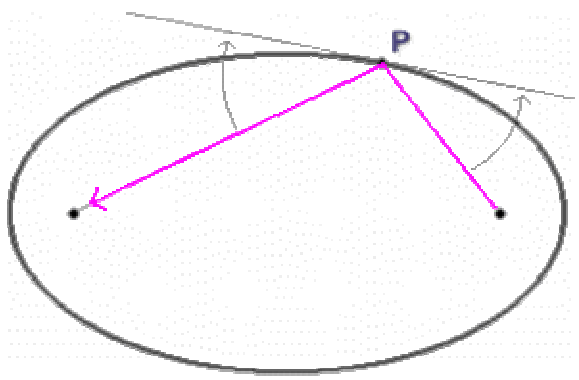
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where *a* and *b* are the lengths of the semimajor and semiminor axes, respectively. (If the major axis of the ellipse is vertical, exchange *a* and *b* in the equation.) The points (*a*, 0) and (-*a*, 0) are called the *vertices* of the ellipse. If the ellipse is translated up/down or left/right, so that its centre is at (*h, k*), then the equation takes the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

If $a = b$, we have the special case of an ellipse whose foci coincide at the centre – that is, a *circle* of radius a.

The ellipse has the following remarkable reflection property. Let P be any point on the ellipse, and construct the line segments joining P to the foci. Then these lines make equal angles to the tangent line at P.



Consequently, any ray emanating from one focus will always reflect off of the inside of the ellipse in such a way as to go straight to the other focus. Architects have exploited this property in many famous buildings. The "whisper chamber" in the United States Capitol is one; stand at one focus and whisper, and anyone at the other focus can hear you with perfect clarity, even though they are much too far away from you to hear a whisper normally. The Mormon Tabernacle in Salt Lake City was also designed as an ellipse (indeed, it is the top half of an ellipsoid), to provide a perfect acoustical environment for choral and organ music.

Geometric Construction

The geometric construction of an ellipse can easily be accomplished with some very simple tools: a piece of string, a pencil, two pins, and a piece of paper. Simply stick the two pieces of string into the piece of paper using the two pins. Pull the string tight (using the pencil) until a triangle is built with the pencil and the two pins as vertices. Now, keeping the string pulled tight, move the pencil around until the ellipse is traced out. (See the enclosed FIGURE E2.)

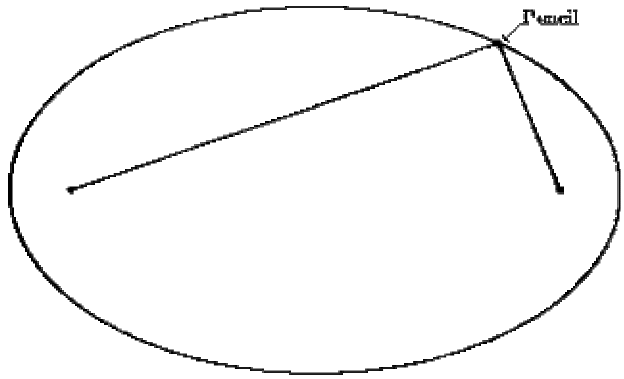


FIGURE E2

Some other terms need to be introduced at this point of the discussion. The line through the foci intersects the ellipse at two points, known as **vertices**. (Vertices is the plural of the term **vertex**.) This line segment joining the vertices is called the **major axis** and its midpoint is called the **centre** of the ellipse. The **minor axis** is the line segment perpendicular to the major axis which also goes through the centre and crosses the ellipse at two points. (By the way, these two points are called minor vertices in the Slovak literature) See FIGURE E3 for a graphical view of some of these key terms.

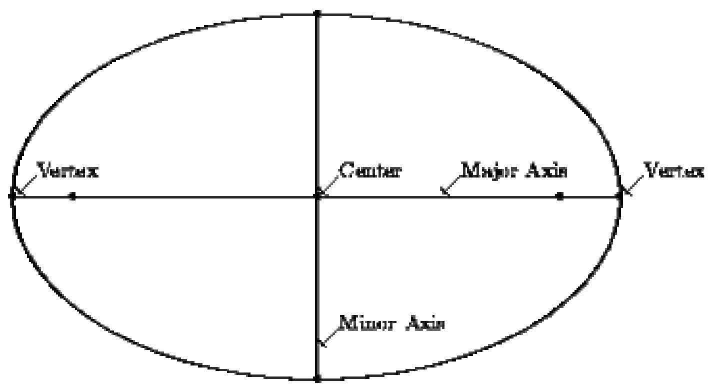


FIGURE E3

<u>Example 1</u>

Consider the equation

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

Given our comments above, this equation yields an ellipse. We see that a = 5 and b = 3 and the graph of this ellipse is the following:

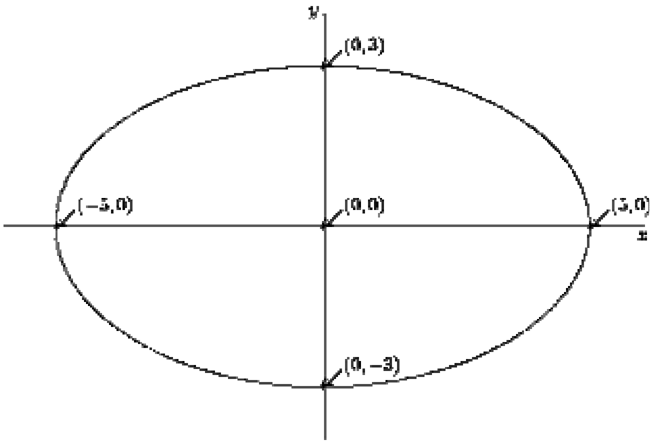


FIGURE E4

Note the relationship of ***a*** and ***b*** to the graph. We see that the value of ***a*** yields half the length of the major axis. Equivalently, we can say that the vertices are found by travelling down the major axis exactly ***a*** units from the centre. We also see that half the length of the minor axis is exactly ***b***.

<u>Example 2</u>

$$\frac{x^2}{9} + \frac{y^2}{25} = 1.$$

It turns out that this ellipse looks very similar to the ellipse in Example 1. The major difference is that this ellipse is now oriented along the vertical axis as opposed to the horizontal. In other words, the major axis of this ellipse is vertical, not horizontal. See FIGURE E5.

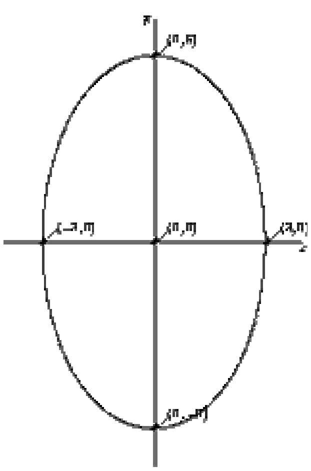


FIGURE E5

<u>Example 3</u>

Find the standard form of the equation of the ellipse with foci at (0, 5) and (0, -5) and with a major axis of length 26.

We must interpret the information given to us in the above problem. First, we see that the foci are 10 units apart (and live on the **y**-axis). Thus, we have **e = 5 and a = 13**, since the length of the major axis is 26. Since the centre of the ellipse is at (0, 0), the vertices of the ellipse must be at (0, 13) and (0, -13). Finally, we just need to find out the value of ***b***. From the relationship $\mathbf{e^2 = a^2 - b^2}$

we know that $\mathbf{5^2 = 13^2 - b^2}$

This can be simplified to $\mathbf{b^2 = 169 - 25 = 144}$

Thus we see that $\mathbf{b = 12}$

Now we can write the standard equation of the ellipse. It is

$$\frac{x^2}{144} + \frac{y^2}{169} = 1.$$

Example 4: Sketch the graph of the ellipse whose equation is

$$\frac{(x-3)^2}{49} + \frac{(y+1)^2}{25} = 1.$$

Again, let's pull as much information out of the equation as possible. We see that the centre of the ellipse is (3, -1). Next, note that

a = 7 and b = 5

Since ***a*** is in the denominator of the term involving the variable ***x***, we know that the major axis of this ellipse is horizontal (parallel with the ***x***-axis). Moreover, we know that the major axis has length 14 and the vertices occur at points which are 7 units in either direction from the centre. This all implies that the vertices are at **(3 + 7, -1)** and **(3 – 7, -1),** which could also be written as **(10, -1)** and **(-4, -1).** As a sidelight, we also know that the endpoints of the minor axis are exactly 5 units above and below the centre, which places them at the points (**3, 4)** and **(3, -6).**

From this information, we can easily plot the ellipse in question. For the sake of completion, let's quickly determine the location of the foci. Again using the relationship

$$c^2 = a^2 - b^2,$$

we know that

$$c^2 = 49 - 25 = 24.$$

Thus, the foci are exactly $\sqrt{24}$ units to the left and right of the centre of the ellipse.

Finally, a sketch of the graph is given in Figure E7.

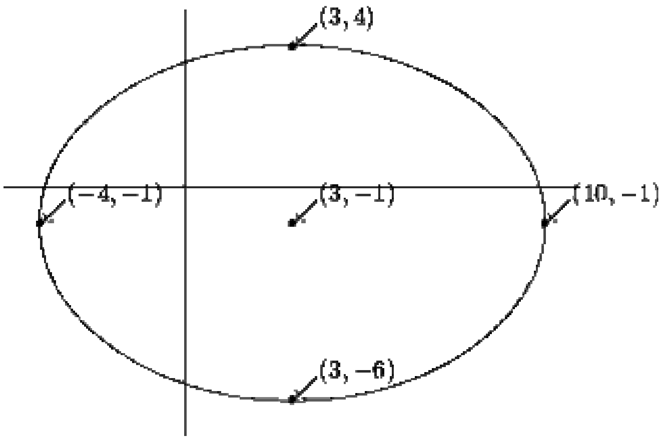


FIGURE E7

Example 5

Sketch the graph of the ellipse whose equation is

$$x^2 + 4y^2 - 6x + 8y + 9 = 0.$$

At this stage of our conics development, we really have not dealt with the equation of a conic in non-standard form. Recall that this is, however, a valid equation for a conic and it happens to be an ellipse. Our first goal is to rewrite this equation into standard form and then to interpret this equation as we sketch the graph. The technique involved in rewriting this equation into standard form is known as "**completing the square**."

We see that

$$x^2 + 4y^2 - 6x + 8y + 9 = 0$$

is equivalent to

$$(x^2 - 6x + \quad) + 4(y^2 + 2y + \quad) = -9.$$

Now we want to fill in the apparent gaps that have been inserted in the parentheses above. This "filling in" is completing the square. We want to write in the number that will make each set of parentheses a perfect square. We do that now:

$$(x^2 - 6x + 9) + 4(y^2 + 2y + 1) = -9 + 9 + 4.$$

Note that, when we add 9 to the left-hand side of the equation, we must also add it to the right-hand side. Also, we are not really adding 1 to the left-hand side; we are really adding 4 since we multiply the 1 by the 4 that is outside the parentheses. Hence, we must also add 4 to the right-hand side.

Rewriting our equation now yields

$$(x - 3)^2 + 4(y + 1)^2 = 4$$

or

$$\frac{(x - 3)^2}{4} + \frac{(y + 1)^2}{1} = 1$$

and we have successfully transformed the equation originally given to us into the standard equation of an ellipse. This ellipse has centre (3, -1) and has values a = 2, b = 1

As noted in previous examples, because ***a*** is in the denominator of the term involving the variable ***x***, we know that the major axis of this ellipse is horizontal (parallel with the ***x***-axis).

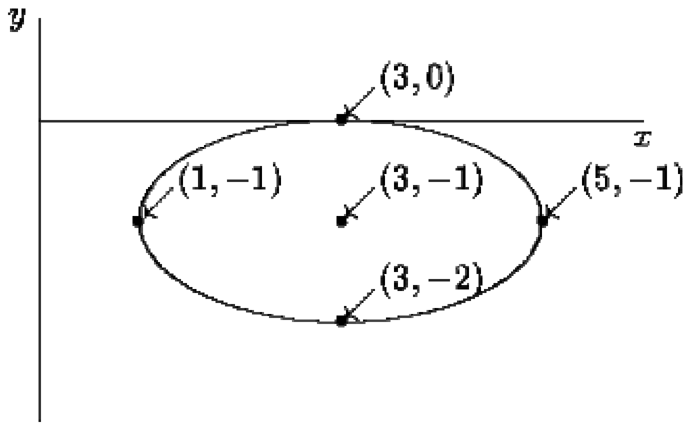
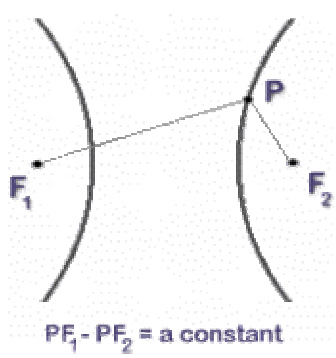


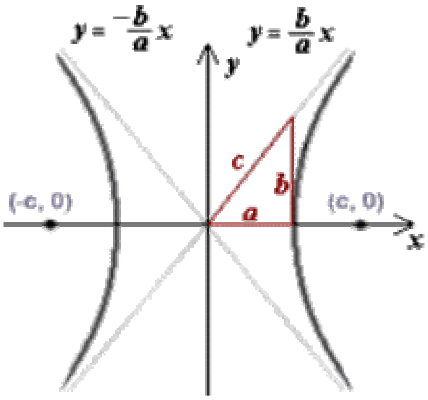
FIGURE E8

HYPERBOLA

The set of all points in the plane, the difference of whose distances from two fixed points, called the foci, remains constant.



Mimicking our procedure with ellipses, we will choose the constant 2*a* to represent the difference of these distances, that is, $PF_1 - PF_2 = 2a$. We will call the two points of the hyperbola which lie on the line connecting the foci the *vertices*, and we then see that the distance between the vertices must be 2*a*. Also, we will call the distance between the foci 2e. Finally, we will define the constant *b* by $e^2 = a^2 + b^2$. (We may do this since evidently e > *a*.) Placing coordinate axes at the centre as before, we obtain this picture:



Applying the distance formulas and substituting for e as we did in the previous cases, we can derive the standard formula of a hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

We note that solving this equation for *y* yields

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2}$$

and letting *x* become arbitrarily large causes this expression to become arbitrarily close to

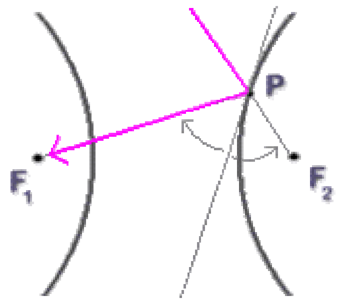
$$y = \pm \frac{b}{a} x$$

Thus we see that the crisscrossing lines in the diagram above are **asymptotes** for the hyperbola, that is, the curve becomes indefinitely close to these lines as the absolute value of *x* grows without bound.

As before, if the principal axis of the hyperbola is vertical instead of horizontal, we switch the roles of *a* and *b*. We may also translate the hyperbola up/down and back/forth, placing the centre at (*h, k*) by modifying our equation thusly:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

The reflection property of the hyperbola is of great importance in optics. Let *P* be any point on one branch of the hyperbola. Then the line segments joining *P* to each of the foci form an angle which is bisected by the tangent line at *P*.



Each hyperbola consists of two **branches**. The line segment which connects the two foci intersects the hyperbola at two points, called the **vertices**. The line segment which ends at these vertices is called the **transverse axis** and the midpoint of this line is called the **centre** of the hyperbola. See Figure H1 for a sketch of a hyperbola with these pieces identified.

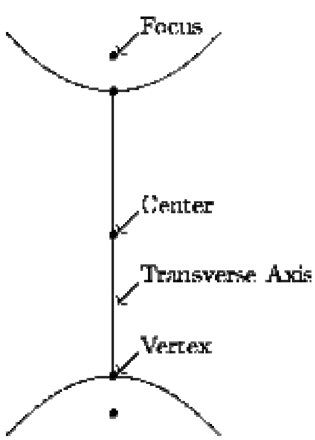


FIGURE H1

Note that, as in the case of the ellipse, a hyperbola can have a vertical or horizontal orientation.

We now turn our attention to the standard equation of a hyperbola. We say that the standard equation of a hyperbola cantered at the origin is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

if the transverse axis is horizontal, or

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

if the transverse axis is vertical.

Notice a very important difference in the notation of the equation of a hyperbola compared to that of the ellipse. We see that ***a*** **always** corresponds to the **positive** term in the equation of the ellipse. The relationship of ***a*** and ***b*** does **not** determine the orientation of the hyperbola. (Recall that the size of ***a*** and ***b*** was used in the section on the ellipse to determine the orientation of the ellipse.) In the case of the hyperbola, the variable in the "positive" term of the equation determines the orientation of the hyperbola. Hence, if the variable ***x*** is in the positive term of the equation, as it is in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

then the hyperbola is oriented as follows:

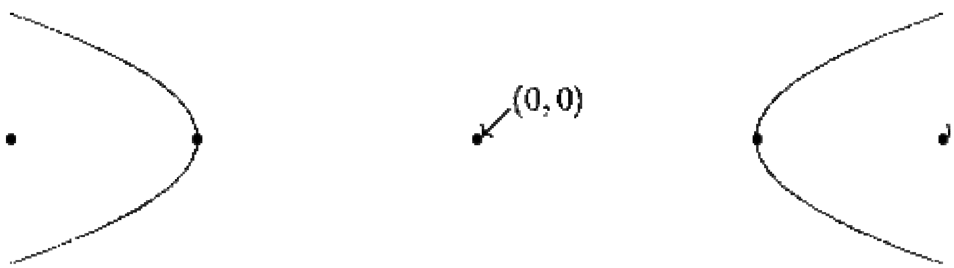


FIGURE H2

If the variable ***y*** is in the positive term of the equation, as it is in the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1,$$

then we see the following type of hyperbola:

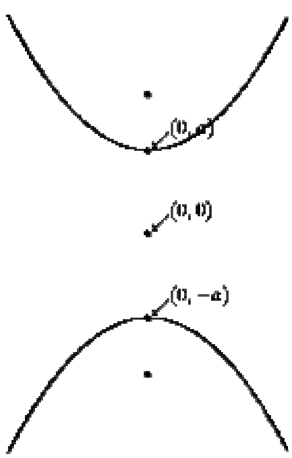


FIGURE H3

Note that the vertices are always ***a*** units from the centre of the hyperbola, and the distance e of the foci from the centre of the hyperbola can be determined using ***a***, ***b***, and the following equality: $e^2 = a^2 + b^2$

We will use this relationship often, so keep it in mind.

The next question you might ask is this: "What happens to the equation if the centre of the hyperbola is **not** (0,0)?" As in the case of the ellipse, if the centre of the hyperbola is (h, k), then the equation of the hyperbola becomes

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

if the transverse axis is horizontal, or

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

if the transverse axis is vertical.

Now in the case of a hyperbola, the distance between the foci is **greater** than the distance between the vertices. Hence, in the case of a hyperbola,

$$e > 1.$$

Recall that for the ellipse,

$$0 \leq e < 1.$$

Two final terms that we must mention are **asymptotes** and the **conjugate axis**. The two branches of a hyperbola are "bounded by" two straight lines, known as *asymptotes*. These asymptotes are easily drawn once one plots the vertices and the points ($\boldsymbol{h}$, $\boldsymbol{k+b}$) and ($\boldsymbol{h}$, $\boldsymbol{k-b}$) and draws the rectangle which goes through these four points. The line segment joining ($\boldsymbol{h}$, $\boldsymbol{k+b}$) and ($\boldsymbol{h}$, $\boldsymbol{k-b}$) is called the *conjugate axis*. The asymptotes then are simply the lines which go through the corners of the rectangle. (See FIGURE H4.)

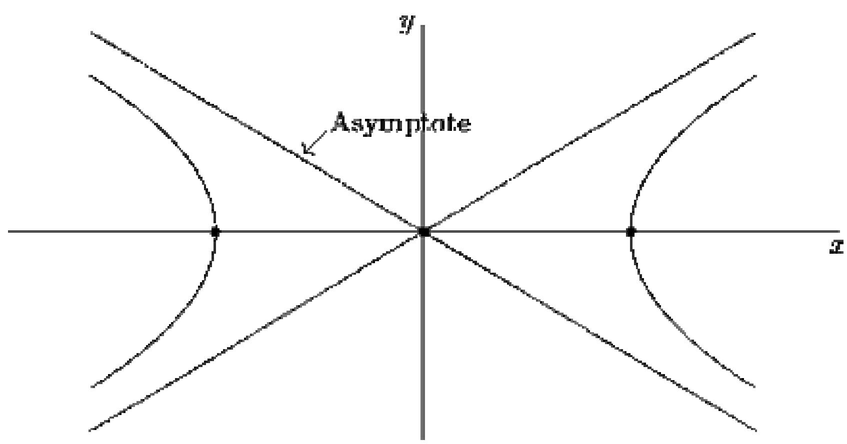


FIGURE H4

Example 1: Consider the equation

$$\frac{x^2}{25} - \frac{y^2}{9} = 1.$$

Given our comments above, this equation yields a hyperbola. (Note the difference between this equation and that in Example 1 of the section on ellipses.) We see that a =5, and b = 3 and the graph of this hyperbola is the following:

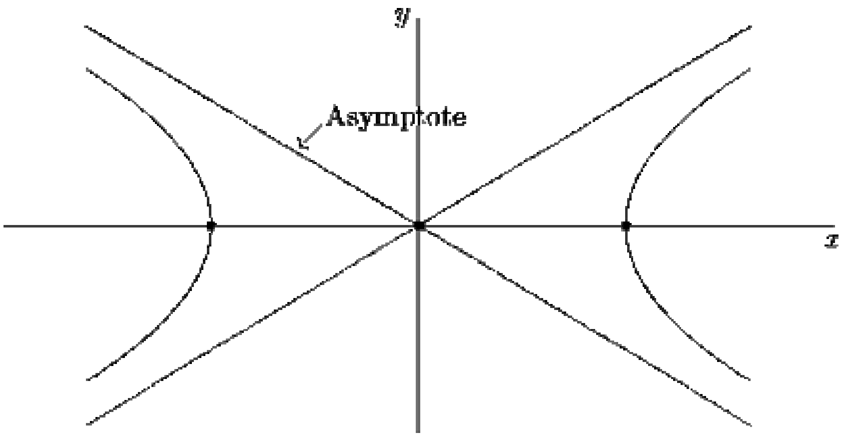


FIGURE H4

Note also, as we finish this example, that the equations of the asymptotes for this hyperbola are

$$y = \frac{3}{5}x$$

and

$$y = -\frac{3}{5}x.$$

Example 2

Consider the equation

$$\frac{x^2}{9} - \frac{y^2}{25} = 1.$$

Note that the only difference between this example and the previous one is that the 9 and 25 have traded places. How does this change the shape of the hyperbola? Is there a change in the orientation from horizontal to vertical? The answer is **no**. Recall that orientation of a hyperbola is **not** determined by the sizes of the denominators in the terms of the standard equation of the hyperbola. Rather, orientation is determined by which variable (**x** or **y**) is in the ``positive'' term. Hence, as is the case in the previous example, this hyperbola is also horizontally oriented. The switch between the 9 and 25 simply changes the shape of the branches. The openings of the branches appear to be wider. See FIGURE H5.

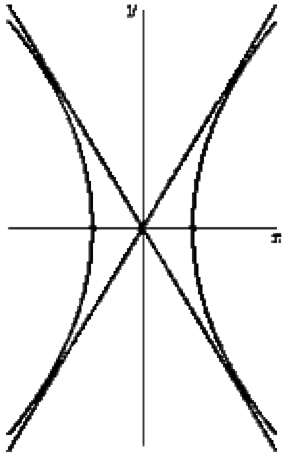


FIGURE H5

Note that difference in shape (between this example and the previous) can also be seen in the equations of the asymptotes. Now we see that

$$a = 3$$

and

$$b = 5.$$

Thus, the slope of the asymptotes is now $\frac{5}{3}$, not $\frac{3}{5}$ as in the previous example.

<u>Example 3</u>

Find the standard form of the equation of the hyperbola with foci at (0, 9) and (0, -9) and transverse axis of length 6.

We must interpret the information given to us in the above problem. First, we see that the foci are 18 units apart (and live on the **y**--axis). Thus, we have e = 9. Moreover, a = 3

since the length of the transverse axis is 6. Since the centre of the hyperbola is at (0, 0), the vertices of the hyperbola must be at (0, 3) and (0, -3). Finally, we just need to find out the value of **b**. From the relationship

$$b^2 = c^2 - a^2,$$

we know that

$$b^2 = 9^2 - 3^2.$$

This can be simplified to

$$b^2 = 81 - 9 = 72.$$

Thus we see that

$$b = \sqrt{72} = 6\sqrt{2}.$$

Now we can write the standard equation of the hyperbola. It is

$$\frac{y^2}{9} - \frac{x^2}{72} = 1.$$

Example 4

Sketch the graph of the hyperbola whose equation is

$$\frac{(x-3)^2}{49} - \frac{(y+1)^2}{25} = 1.$$

Again, let's pull as much information out of the equation as possible. We see that the centre of the hyperbola is (3, -1). Next, note that a = 7 and b = 5.

Since the ``positive'' term in the equation involves the variable **x**, we know that the transverse axis of this hyperbola is horizontal (parallel with the **x**--axis). Moreover, we know that the transverse axis has length 14 and the vertices occur at points which are 7 units in either direction of the centre. This all implies that the vertices are at (3 + 7, -1) and (3 – 7, -1), which could also be written as (10, -1) and (-4, -1). As a sidelight, we also know that the endpoints of the conjugate axis are exactly 5 units above and below the centre, which places them at the points (3, 4) and (3, -6).

From this information, we can easily plot the hyperbola in question. For the sake of completion, let's quickly determine the location of the foci. Again using the relationship

$$b^2 = c^2 - a^2,$$

we know that

$$c^2 = 49 + 25 = 74.$$

Thus, the foci are exactly $\sqrt{74}$ units to the left and right of the centre of the hyperbola.

Finally, a sketch of the graph is given in Figure H7.

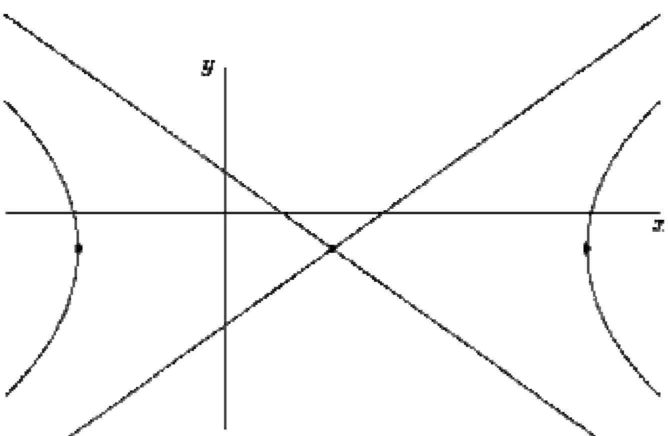


FIGURE H7

Example 5

Sketch the graph of the hyperbola whose equation is

$$y^2 - 4x^2 - 6y - 8x - 11 = 0.$$

Note that this is a valid equation for a hyperbola, even though it is not in **standard** form. Our first goal is to rewrite this equation into standard form and then to interpret this equation as we sketch the graph. As we have seen in previous work, we need to use the technique of ``completing the square'' to work this out.

We see that

$$y^2 - 4x^2 - 6y - 8x - 11 = 0$$

is equivalent to

$$(y^2 - 6y + \quad) - 4(x^2 + 2x + \quad) = 11.$$

Now we want to fill in the apparent gaps that have been inserted in the parentheses above. This ``filling in'' is completing the square. We want to write in the number that will make each set of parentheses a perfect square. We do that now:

$$(y^2 - 6y + 9) - 4(x^2 + 2x + 1) = 11 + 9 - 4.$$

Note that, when we add 9 to the left--hand side of the equation, we must also add it to the right--hand side. Also, we are not really adding 1 to the left--hand side; we are really subtracting 4 since we multiply the 1 by the **-4** that is outside the parentheses. Hence, we must also subtract 4 from the right--hand side.

Rewriting our equation now yields

$$(y - 3)^2 - 4(x + 1)^2 = 16$$

or

$$\frac{(y-3)^2}{16} - \frac{(x+1)^2}{4} = 1$$

and we have successfully transformed the equation originally given to us into the standard equation of a hyperbola. This hyperbola has centre (3, -1) and has values a = 4 and b = 2.

Since the variable **y** is in the ``positive'' term here, we know that the transverse axis of this hyperbola is vertical (parallel with the **y**--axis).

The graph of this hyperbola is given in Figure H8.

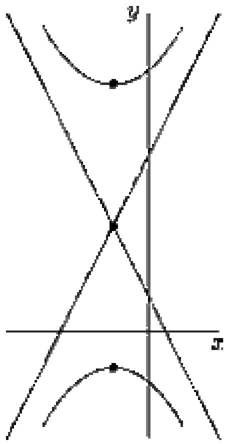
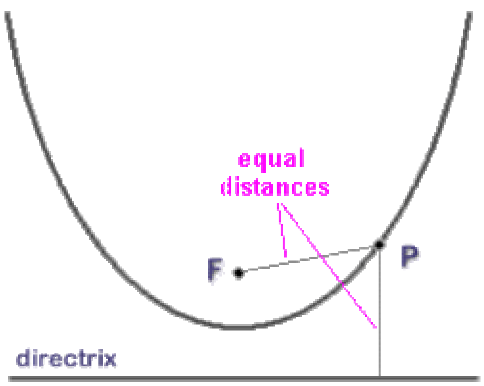


FIGURE HB

PARABOLA

The set of all points in the plane whose distances from a fixed point, called the **focus**, and a fixed line, called the **directrix**, are always equal.



The point directly between – and hence closest to – the focus and the directrix is called the ***vertex*** of the parabola.

To derive the equation of a parabola in rectangular coordinates, we again choose a convenient location for the axes, placing the origin at the vertex so that the *y*-axis is the axis of symmetry. We denote the distance from the focus to the directrix by *p* (called parameter).

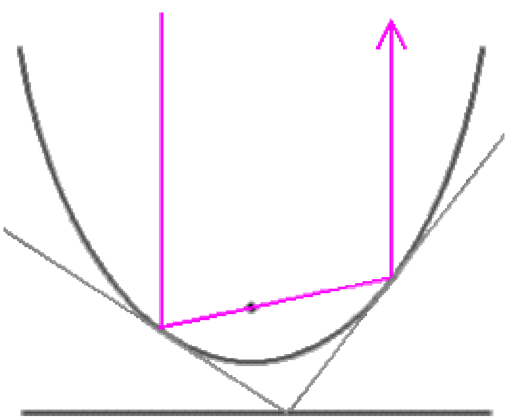
Then distance from the vertex to the focus and to the directrix is the same, i.e. $\frac{p}{2}$.

Then the standard equation of a parabola is $y^2 = \pm 2px$ (when axis is coincident with x – axis)

or $x^2 = \pm 2py$ (when axis is coincident with y- axis).

This one is the equation of a parabola opening upwards, with its vertex at the origin. If we introduce a negative sign, we get a parabola opening downwards. If we interchange the roles of *x* and *y*, we get a parabola opening to the right (or to the left if there is a negative). We may translate the parabola up/down or back/forth, putting the vertex at the point (*h, k*) if we write our equation as $(x - h)^2 = 2\ p\ (y - k)$

The reflection property of parabolas is very important because it has so many practical uses. Let *P* be any point on the parabola. Construct the line segment joining *P* to the focus, and a ray through *P* that is parallel to the axis of symmetry. The line segment and ray will always make equal angles to the tangent line at *P*. Consequently, any ray emanating from the focus will reflect off of the parabola so as to point directly outwards, parallel to the axis. This property is made use of in the design of flashlights, headlights, and spotlights, for instance. Conversely, any ray entering the parabola that is parallel to the axis will be reflected to the focus. This property is exploited in the design of radio and satellite receiving dishes, and solar collectors.



The reflection property of parabolas is related to the curious property that the tangent lines at the endpoints of any chord through the focus (as shown above) intersect on the directrix, and always do so in a right angle.

Parabolas are also important in the study of ballistics, the movement of a body under the force of gravity.

Note that the graph of a parabola is similar to one branch of a hyperbola. However, you should realize that a parabola is **not** simply one branch of a hyperbola. Indeed, the branches of a hyperbola approach linear asymptotes, while a parabola does not do so.

Several other terms exist which are associated with a parabola. The midpoint between the focus and directrix of the parabola is called the **vertex** and the line passing through the focus and vertex is called the **axis** of the parabola. (This is similar to the major axis of the ellipse and the transverse axis of the hyperbola.) See Figure P2.

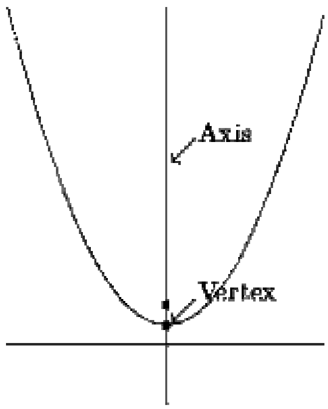


FIGURE P2

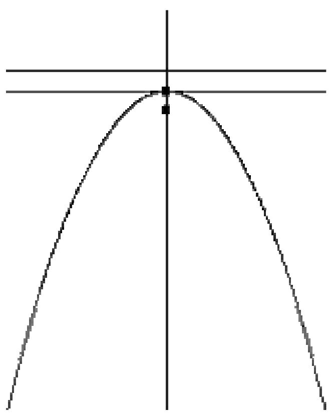


FIGURE P5

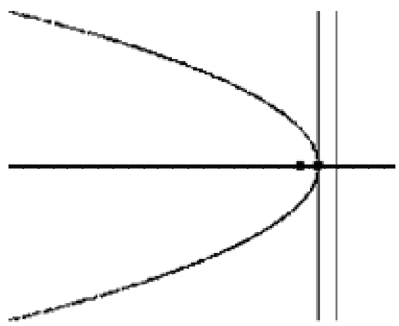


FIGURE P6

Thus, we see that there are four different orientations of parabolas, which depend on a) which variable is squared (***x*** or ***y***) and b) whether ***d*** is positive or negative.

<u>Example 1</u>

Consider the equation

$$x^2 = 8y.$$

Given our comments above, this equation yields a parabola. (Note that this equation only has one square term. This indicates that we have a parabola here, as opposed to an ellipse or hyperbola.)

The value of **p** in this case is fairly clear; namely, p = 4

The vertex of this parabola is at (0, 0) and the axis of this parabola is vertical (since the variable **x** is the squared variable.) Thus, the focus of this parabola is situated at the point (0, 2) and the directrix is the horizontal line 4 units below the vertex. Thus, the equation of the directrix is given by

$$y = -2.$$

See Figure P7 for a sketch of this parabola.

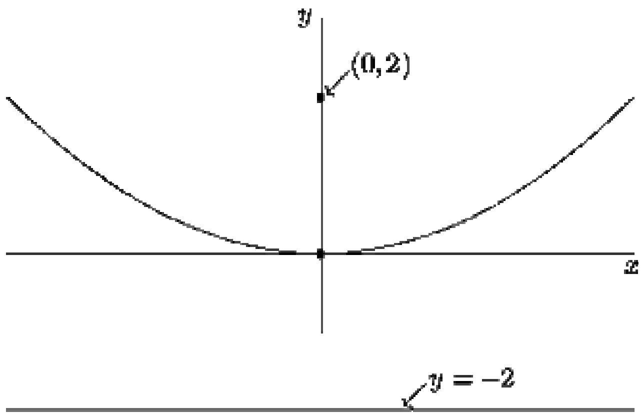


FIGURE P7

Example 2

$$y^2 = -16x.$$

Again, realize that this is a parabola (as opposed to an ellipse or a hyperbola) because it only has one squared term (as opposed to two). Because the **y** is squared instead of the **x**, we automatically know that this parabola has a horizontal axis. The only other question dealing with orientation is whether the parabola opens to the left or to the right.

We note here that p = -8.

This indicates that the parabola will indeed open to the left. The vertex is again (0, 0), as in the first example, and the focus will now occur at the point (-4, 0). Finally, the equation of the directrix is x = 4.

See Figure P8 for the graph of this parabola.

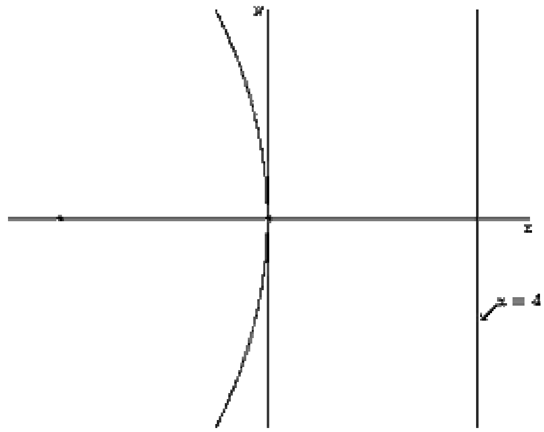


FIGURE P8

Example 3

Consider the equation

$$(x-3)^2 = 24(y-5).$$

This is the standard equation of the parabola with vertex (3, 5) and p = 12.

Hence, because the axis is vertical, we know that the focus is exactly 6 units above the vertex, placing it at (3, 11). A sketch of this parabola can be found in Figure P9.

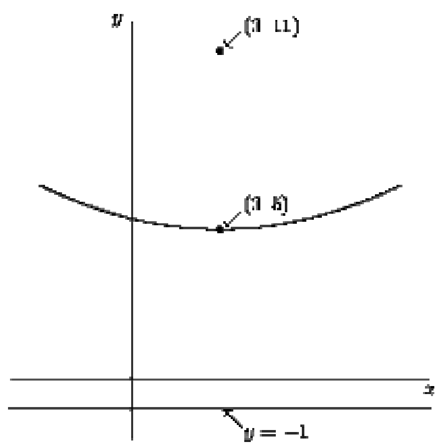


FIGURE P9

Example 4

Find the standard form of the equation of the parabola with vertex (2, 1) and focus (5, 1).

We see several things from the information given. First, note that the axis of this parabola is oriented horizontally. Next, note that p = 6.

Thus, we can quickly write down the standard equation of this parabola. It is

$$(y-1)^2 = 12(x-2).$$

See Figure P10 for a sketch of this parabola.

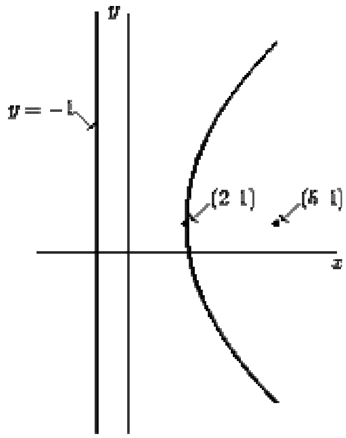


FIGURE P10

<u>Example 5</u>

Determine the vertex, focus, and directrix of the parabola given by the equation

$$y^2 + 6y + 8x = -25.$$

In order to accomplish this task, we need to rewrite the equation in standard form. This will again involve completing the square. We see that this is accomplished by the following:

$$(y^2 + 6y + 9) + 8x = -25 + 9.$$

so

$$(y + 3)^2 = -16 - 8x.$$

Factoring out **-8** from the right--hand side leaves

$$(y + 3)^2 = -8(x + 2).$$

We have now rewritten our equation into standard form. Note that the vertex is (-2, -3) and p = -4.

We also see that the axis is horizontal and that the parabola opens to the left (since **p** is negative.) Thus, the directrix is the vertical line that lies 2 units to the right of the vertex, yielding the equation x = 0.

Finally, the focus is the point 2 units to the right of the vertex, which in this case is the point $(-4, -3)$.