PURE IMAGINARY NUMBERS

The square root of a negative number (i.e., $\sqrt{-1}, \sqrt{-5}$) is called a **pure imaginary number**. Since by definition $\sqrt{-5} = \sqrt{5} \cdot \sqrt{-1}$ it is convenient to introduce the symbol i = $\sqrt{-1}$ and to adopt $\sqrt{-5} = i\sqrt{5}$ as the standard form for these numbers.

The symbol i has the property $i^2 = -1$ and for higher integral powers we have
$i^3 = i^2 \cdot i = (-1)i = -i$
$i^4 = i^2 \cdot i^2 = 1$ etc.
The use of the standard form simplifies the operation on pure imaginaries and eliminates the possibility of certain common errors. Thus,
$\sqrt{-9} \cdot \sqrt{4} = \sqrt{-36} = 6i$, but $\sqrt{-9} \cdot \sqrt{-4} = (3i)(2i) = 6i^2 = -6$

Notice the cyclic nature of the powers of i. $i^{4k+1} = i,\ i^{4k+2} = -1,\ i^{4k+3} = -i,\ i^{4k+4} = 1$ for every natural number k.

COMPLEX NUMBERS

A number $a + bi$, where a and b are real numbers, is called a complex number. The first term a is called the real part of the complex number and the second term bi is called the pure imaginary part.
Complex numbers may be thought of as including all real numbers and all pure imaginary numbers. For example $5 = 5 + 0i$ and $3i = 0 + 3i.$
Two complex numbers $a + bi$ and $c + di$ are said to be equal if and only if $a = c$ and $b = d$.
The conjugate of a complex number $a + bi$ is the complex number $a - bi$ .Thus, $2 + 3i$ and $2 - 3i$ is a pair of conjugate complex numbers.

ALGEBRAIC OPERATIONS

(1) ADDITION. To add complex numbers, add the real parts and add the pure imaginary parts. EXAMPLE 1: $(2+3i)+(4-5i) = (2+4)+i\ (3-5) = 6-2i$

(2) SUBTRACTION. To subtract two complex numbers, subtract the real parts and subtract the pure imaginary parts. EXAMPLE 2: $(2+3i)-(4-5i) = (2-4)+i\ (3-(-5)) = -2+8i$

(3) MULTIPLICATION. To multiply two complex numbers, carry out the multiplication as if the numbers were ordinary binomials and replace i^2 by -1.
EXAMPLE 3: $(2+3i)(4-5i) = 8+2i-15i^2 = 8+2i-15(-1) = 23+2i$

(4) DIVISION. To divide two complex numbers, multiply both numerator and denominator of the fraction by the conjugate of the denominator.
EXAMPLE 4: $\dfrac{2+3i}{4-5i} = \dfrac{(2+3i)(4+5i)}{(4-5i)(4+5i)} = \dfrac{(8-15)+(10+12)\ i}{16+25} = \dfrac{7}{41} + \dfrac{22}{41}\ i.$

Note the form of the result; it is neither $\dfrac{-7+22\ i}{41}$ nor $\dfrac{1}{41}(-7+22i)$.

GRAPHIC REPRESENTATION OF COMPLEX NUMBERS

The complex number $x + yi$ may be represented graphically by the point P (see Fig. 1) whose rectangular coordinates are $[x, y]$. The point O, having coordinates $[0,0]$, represents the complex number $0 + 0i = 0$. All points on the x axis have coordinates of the form $[x,0]$ and correspond to real numbers $x + 0i = x.$ For this reason; the x axis is called **the axis of reals**. All points on the y axis have coordinates of the form $[0, y]$ and correspond to pure imaginary numbers $0 + yi = yi$ .The y axis is called the axis of imaginaries. The plane on which the complex numbers are represented is called the complex plane. See Fig. 1. In addition to representing a complex number by a point P in the complex plane, the number may be represented by the directed line segment or vector **OP**. See Fig. 2.The vector OP is sometimes denoted by $\overrightarrow{OP}$ and is the directed line segment beginning at O and terminating at P.

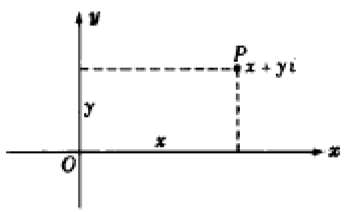


Fig. 1

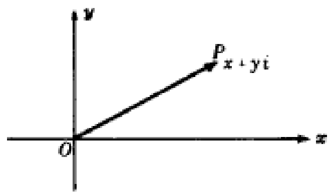


Fig. 2

GRAPHIC REPRESENTATION OF ADDITION AND SUBTRACTION.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers. The vector representation of these numbers suggests the illustrated parallelogram law for determining graphically the sum $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$, since the coordinates of the endpoint of the vector $z_1 + z_2$ must be, for each of the x coordinates and the y coordinates the sum of the corresponding x or y values. See Fig. 3.
Since $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 + iy_1) + (-x_2 - iy_2)$, the difference $z_1 - z_2$ of the two complex numbers may be obtained graphically by applying the parallelogram law to $x_1 + iy_1$ and $-x_2 - iy_2$.(See Fig. 4.)
In Fig. 5 both the sum OR $= z_1 + z_2$ and the difference OS $= z_1 - z_2$ are shown. Note that the segments **OS** and P_2P_1 (the other diagonal of OP_2RP_1) are congruent.

Fig. 3

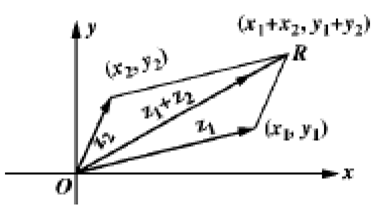


Fig. 4

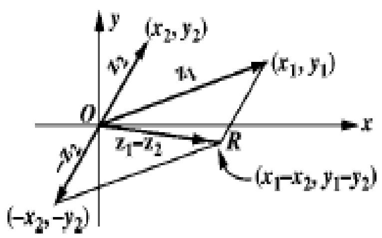
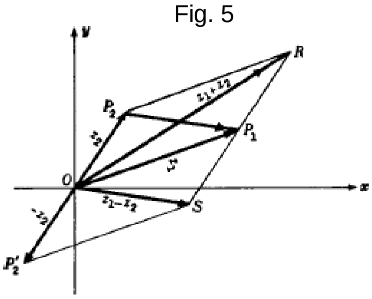


Fig. 5



POLAR OR TRIGONOMETRIC FORM OF COMPLEX NUMBERS

Let the complex number $x + yi$ be represented (Fig. 6) by the vector OP. This vector (and hence the complex number) may be described in terms of the length r of the vector and any positive angle y which the vector makes with the positive x axis (axis of positive reals).The number $r = \sqrt{x^2 + y^2}$ is called **the modulus** or **absolute value of the complex number**. The angle f, called **the amplitude of the complex number**, is usually chosen as the smallest, positive angle for which $\tan f = \frac{y}{x}$ but at times it will be found more convenient to choose some other angle coterminal with it.

From Fig. 6, $x = r\cos f$; and $y = r\sin f$; then $z = x + iy = r\cos f + ir\sin f = r(\cos f + i\sin f)$.

We call $z = r(\cos f + i\sin f)$ **the polar or trigonometric form** and $z = x + iy$ **the rectangular form** of the complex number z.

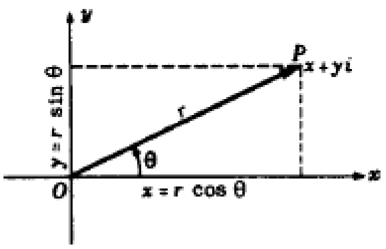


Fig. 6

EXAMPLE 5:

Express $z = 1 - i\sqrt{3}$ in polar form.

The modulus is $r = \sqrt{(1)^2 + (-3)^2} = 2$

Since $\tan f = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$, the amplitude f is either 120° or 300°

.Now we know that P lies in quadrant IV; hence, $f = 300^\circ$ and the required polar form is

$z = r(\cos f + i\sin f) = 2(\cos 300^\circ + i\sin 300^\circ)$

Note that z may also be represented in polar form by

$z = 2[\cos(300^\circ + n360^\circ) + i\sin(300^\circ + n360^\circ)]$ where n is any integer.

EXAMPLE 6:

Express the complex number $z = 8(\cos 210^\circ + i\sin 210^\circ)$ in rectangular form.

Since $\cos 210^\circ = \frac{-\sqrt{3}}{2}$ and $\sin 210^\circ = \frac{-1}{2}$

$z = 8(\cos 210^\circ + i\sin 210^\circ) = 8\left[\frac{-\sqrt{3}}{2} + i\left(\frac{-1}{2}\right)\right] = -4\sqrt{3} - 4i$ is required rectangular form.

MULTIPLICATION AND DIVISION IN POLAR FORM

MULTIPLICATION.
The modulus of the product of two complex numbers is the product of their moduli, and the amplitude of the product is the sum of their amplitudes.

DIVISION.
The modulus of the quotient of two complex numbers is the modulus of the dividend divided by the modulus of the divisor, and the amplitude of the quotient is the amplitude of the dividend minus the amplitude of the divisor.

EXAMPLE 7:

Find (a) the product z_1z_2, (b) the quotient $\frac{z_1}{z_2}$, and (c) the quotient $\frac{z_2}{z_1}$ where

$z_1 = 2(\cos 300^\circ + i\sin 300^\circ)$ and $z_2 = 8(\cos 210^\circ + i\sin 210^\circ)$

(a) The modulus of the product is 2(8) = 16.
The amplitude is $300^\circ + 210^\circ = 510^\circ$, but following the convention, we shall use the smallest positive coterminal angle $510^\circ - 360^\circ = 150^\circ$.
Thus, $z_1z_2 = 16(\cos 150^\circ + i\sin 150^\circ)$

(b) The modulus of the quotient $\frac{z_1}{z_2}$ is $\frac{2}{8} = \frac{1}{4}$

and the amplitude is $300^\circ - 210^\circ = 90^\circ$.

Thus, $\frac{z_1}{z_2} = \frac{1}{4}(\cos 90^\circ + i\sin 90^\circ)$

(c) The modulus of the quotient $\frac{z_2}{z_1}$ is $\frac{8}{2} = 4$

The amplitude is $210^\circ - 300^\circ = -90^\circ$, but we shall use the smallest positive coterminal angle $-90^\circ + 360^\circ = 270^\circ$.

Thus $\frac{z_2}{z_1} = 4(\cos 270^\circ + i\sin 270^\circ)$.

NOTE: From Examples 5 and 6 the numbers are $z_1 = 1 - i\sqrt{3}$ and $z_2 = -4\sqrt{3} - 4i$ in rectangular form. THEN,

$z_1z_2 = (1 - i\sqrt{3})(-4\sqrt{3} - 4i) = -8\sqrt{3} + 8i = 16(\cos 150^\circ + i\sin 150^\circ)$

and

$$\frac{z_1}{z_2} = \frac{1 - i\sqrt{3}}{-4\sqrt{3} - 4i} \cdot \frac{-4\sqrt{3} + 4i}{-4\sqrt{3} + 4i} = \frac{-4\sqrt{3} + 4i + 12i + 4\sqrt{3}}{64} = \frac{16i}{64} = \frac{1}{4}i = \frac{1}{4}(\cos 90^\circ + i\sin 90^\circ)$$

DE MOIVRE'S THEOREM.

If n is any rational number, $\left[r(\cos f + i\sin f)\right]^n = r^n(\cos nf + i\sin nf)$
A proof of this theorem is a verification for n = 2 and n = 3
Let $z = r(\cos f + i\sin f)$
For n = 2
$z^2 = z \cdot z = \left[r(\cos f + i\sin f)\right]\left[r(\cos f + i\sin f)\right] = r^2\left[(\cos^2 f - \sin^2 f) + i(2\sin f\cos f)\right] = r^2(\cos 2f + i\sin 2f)$
For n = 3
$z^3 = z^2 \cdot z = r^2(\cos 2f + i\sin 2f)\cdot\left[r(\cos f + i\sin f)\right] = r^3\left[(\cos 2f\cos f - \sin 2f\sin f) + i(\sin 2f\cos f + \cos 2f\sin f)\right] =$
$= r^3(\cos 3f + i\sin 3f)$

EXAMPLE 8:

$$\left(\sqrt{3} - i\right)^{10} = \left[2(\cos 330^\circ + i\sin 330^\circ)\right]^{10} = 2^{10}(\cos 10\cdot 330^\circ + i\sin 10\cdot 330^\circ) =$$
$$= 1024(\cos 60^\circ + i\sin 60^\circ) = 1024\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 512 + 512i\sqrt{3}$$

ROOTS OF COMPLEX NUMBERS

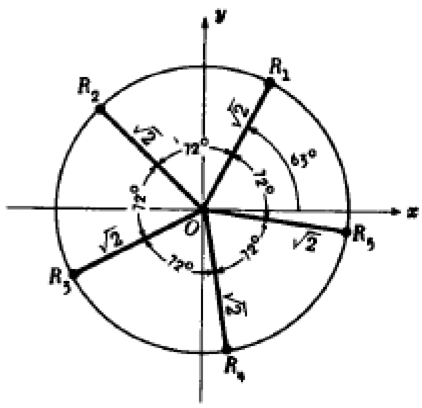
We state, without proof, the theorem:
A complex number $a + bi = r(\cos f + i\sin f)$ has exactly n distinct nth roots.
The procedure for determining these roots is given in Example 9.

EXAMPLE 9: Find all fifth roots of $4 - 4i$.
The usual polar form of $4 - 4i = 4\sqrt{2}(\cos 315^\circ + i\sin 315^\circ)$, but we shall need the more general form $z = 4\sqrt{2}\left[\cos(315^\circ + k360^\circ) + i\sin(315^\circ + k360^\circ)\right]$, where k is any integer, including zero.

Using De Moiver's theorem, a fifth root of $4 - 4i$ is given by

$$\left[4\sqrt{2}\left[\cos(315^\circ + k360^\circ) + i\sin(315^\circ + k360^\circ)\right]\right]^{\frac{1}{5}} = \left(4\sqrt{2}\right)^{\frac{1}{5}}\left(\cos\frac{315^\circ + k360^\circ}{5} + i\sin\frac{315^\circ + k360^\circ}{5}\right) =$$
$$= \sqrt{2}\left[\cos(63^\circ + k72^\circ) + i\sin(63^\circ + k72^\circ)\right]$$

Assigning in turn the values k = 0; 1; . . ., we find



k = 0: $\sqrt{2}\left[\cos 63^\circ + i\sin 63^\circ\right] = R_1$
k = 1: $\sqrt{2}\left[\cos 135^\circ + i\sin 135^\circ\right] = R_2$
k = 2: $\sqrt{2}\left[\cos 207^\circ + i\sin 207^\circ\right] = R_3$
k = 3: $\sqrt{2}\left[\cos 279^\circ + i\sin 279^\circ\right] = R_4$
k = 4: $\sqrt{2}\left[\cos 351^\circ + i\sin 351^\circ\right] = R_5$
k = 5: $\sqrt{2}\left[\cos 423^\circ + i\sin 423^\circ\right] = \sqrt{2}\left[\cos 63^\circ + i\sin 63^\circ\right] = R_1$, etc:
Thus, the five fifth roots are obtained by assigning the values 0; 1; 2; 3; 4 (i.e.; 0; 1; 2; 3; . . . ; n-1) to k.
The modulus of each of the roots is; hence these roots lie on a circle of radius with centre at the origin. The difference in amplitude of two consecutive roots is 72°; hence the roots are equally spaced on this circle, as shown in Fig. 8.

Exercise 1

1, Perform the indicated operations, simplify, and write the results in the form a+bi.
a) $(3-4i)+(-5+7i)=$
b) $(4+2i)-(-1+3i)=$
c) $(2+i)(3-2i)=$
d) $(3+4i)(3-4i)=$
e) $\dfrac{3-2i}{2-3i}=$
f) $\dfrac{1+3i}{2+i}=$

2, Find x and y such that 2x – yi = 4+3i

3, Represent graphically (as a vector) the following complex numbers:
a) 3 + 2i
b) 2 – i
c) -2 + i
d) -1 – 3i

4, Express each of following complex number z in polar form (trigonometric form)
a) -1 + i$\sqrt{3}$
b) 6$\sqrt{3}$ + 6i
c) 2 - 2i
d) -3
e) 4i
f) -3 – 4i

5, Express each of following complex numbers z in rectangular form
a) $4(\cos 240^\circ + i\sin 240^\circ)$
b) $2(\cos 315^\circ + i\sin 315^\circ)$
c) $3(\cos 90^\circ + i\sin 90^\circ)$
d) $5(\cos 128^\circ + i\sin 128^\circ)$

6, Perform the indicated operations, giving the results in both forms
a) $5(\cos 170^\circ + i\sin 170^\circ)\mathrm{x}(\cos 55^\circ + i\sin 55^\circ)$
b) $2(\cos 50^\circ + i\sin 50^\circ)\mathrm{x}3(\cos 40^\circ + i\sin 40^\circ)$
c) $6(\cos 110^\circ + i\sin 110^\circ)\mathrm{x}\dfrac{1}{2}(\cos 212^\circ + i\sin 212^\circ)$
d) $10(\cos 305^\circ + i\sin 305^\circ)\div 2(\cos 65^\circ + i\sin 65^\circ)$
e) $4(\cos 220^\circ + i\sin 220^\circ)\div 2(\cos 40^\circ + i\sin 40^\circ)$
f) $6(\cos 230^\circ + i\sin 230^\circ)\div 3(\cos 75^\circ + i\sin 75^\circ)$

7, Express each of following complex number z in polar form (trigonometric form), perform the indicated operation, and give the result in rectangular form

a) $\left(-1+i\sqrt{3}\right)\left(\sqrt{3}+i\right)$

b) $\left(3-i3\sqrt{3}\right)\left(-2-i2\sqrt{3}\right)$

c) $\left(4-i4\sqrt{3}\right)\div\left(-2\sqrt{3}+2i\right)$

d) $-2\div\left(-\sqrt{3}+i\right)$

e) $6i\div(-3-3i)$

f) $\left(1+i\sqrt{3}\right)\left(1+i\sqrt{3}\right)$

g) $(3+2i)(2+i)$

h) $(2+3i)\div(2-3i)$

Exercise 2

1, Evaluate each of the following using De Moivre`s theorem and express each result in rectangular form:

a) $\left(1+i\sqrt{3}\right)^4$

b) $\left(\sqrt{3}-i\right)^5$

c) $(-1+i)^{10}$

d) $(2+3i)^4$

2, Find the indicated roots in rectangular form if possible.

a) square roots of $2-i2\sqrt{3}$

b) Fourth roots of $-8-i8\sqrt{3}$

c) Cube roots of $-4\sqrt{2}+i4\sqrt{2}$

d) Cube roots of 1

e) Fourth roots of i

f) Sixth roots of -1

g) Fourth roots of -16i

h) Fifth roots of $1+3i$