

COMPLEX NUMBERS C

The set of complex numbers is in fact the set of real numbers extended with the imaginary unit

$i = \sqrt{-1}$. A complex number is every number of the form $z = a + bi$, where a and $b \in \mathbb{R}$.

We are now extending ideas of number further, since the equation $x^2 = -16$ has no solution in terms of real numbers. We introduce, therefore, the concept of an imaginary number $i = \sqrt{-1}$. Thus now, the given equation has two solutions $x = 4i$, or $x = -4i$.

REAL NUMBERS VERSUS COMPLEX NUMBERS

A complex number $z = a + bi$ where a and b are real numbers and i is the imaginary unit is in its algebraic form. Here, a is the real part of the complex number and bi is the imaginary part. Important information is that when the imaginary part (bi) equals 0, the number is just a real number (a).

Therefore each real number can be rewritten into the form of a complex number with imaginary part which equals zero (e.g.: $4+0i$). And so we can say that the set of all real numbers (\mathbb{R}) is the subset of the set of all complex numbers (\mathbb{C}).

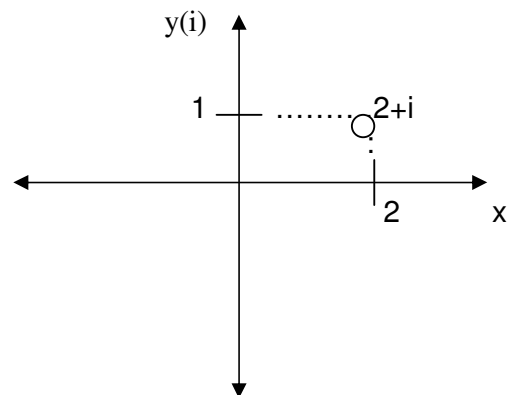
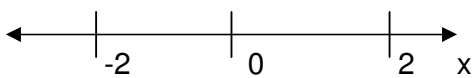
Two complex numbers are equal if, and only if, their real parts are equal and their imaginary parts are equal.

Thus $a + bi = c + di \Leftrightarrow a = c$ and $b = d$.

Complex numbers can be added, subtracted, multiplied and divided as well as real numbers applying the commutative, distributive and associative laws, but they also have some additional properties. One of the examples can be that real numbers themselves do not provide a solution for every polynomial algebraic equation with real coefficients, while complex numbers do (as can be seen in the beginning).

Another very important difference is the representation of real numbers and complex numbers.

Real numbers are represented just on one axis (x-axis) as shown on the left, while complex numbers are represented on 2 different axes (x-axis and y(i)-axis) as shown on the right. The real part is represented on the x-axis, the imaginary one on the y-axis.



REPRESENTING COMPLEX NUMBERS IN A COMPLEX PLANE

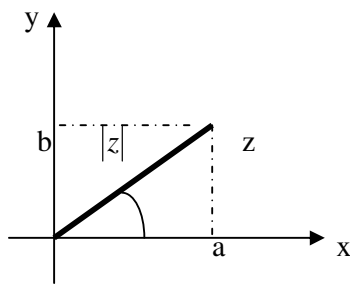
Complex numbers can be represented geometrically using the x and y axes as the Real and Imaginary axes. The plane of the axes is then referred to as the complex plane and a diagram showing a complex number is said to be an Argand diagram.

On the Argand diagram, each complex number is represented by a line of a certain length in a particular direction. Thus each complex number is shown as a vector on the Argand director.

Plotting complex numbers is very similar to plotting a point (x, y) in rectangular coordinates. The difference is that the y-axis now represents imaginary numbers. Thus each complex number will have two ‘coordinates’ [a, b].

From the following picture we can find out the *modulus* of a complex number as well as its *argument*.

The **modulus** is the length of the line connecting the complex number to the origin. The modulus can easily be calculated by the Pythagorean Theorem.



$$|z|^2 = a^2 + b^2$$

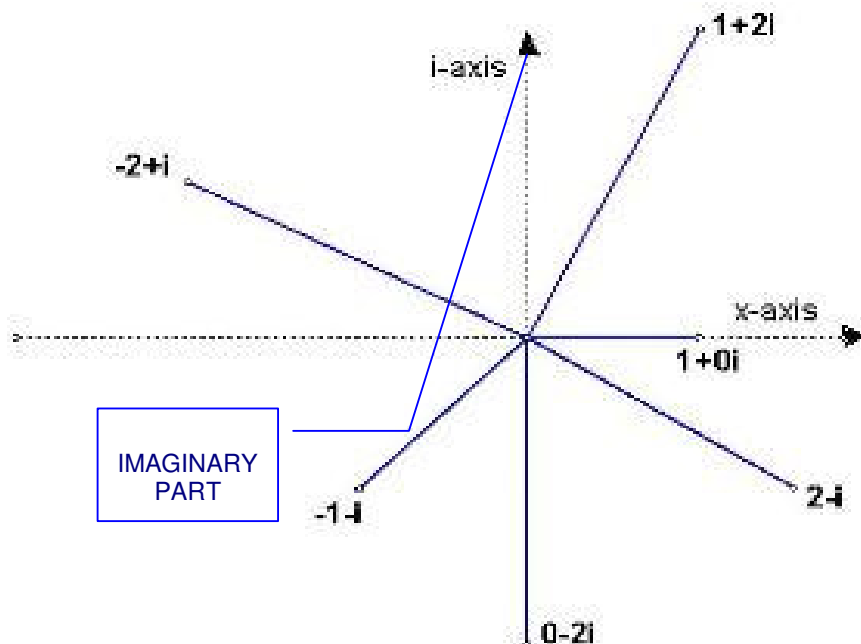
$$|z| = \sqrt{a^2 + b^2}$$

$$\sin \varphi = \frac{b}{|z|}$$

$$\cos \varphi = \frac{a}{|z|}$$

The **argument** is the angle between the line and x-axis, when measured from the positive part of x-axis. The argument can be calculated using the trigonometric identity that:

theta = inverse tangent of (i-component) over (x-component), i.e. $\text{tg } \theta = \frac{iy}{x} = \frac{b}{a}$.



BASIC OPERATIONS WITH COMPLEX NUMBERS

Complex numbers are added (or subtracted) by adding (or subtracting) their real part and also their imaginary parts.

ADDITION

Addition of two complex numbers $a+bi$ & $c+di$ is defined as follows:

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

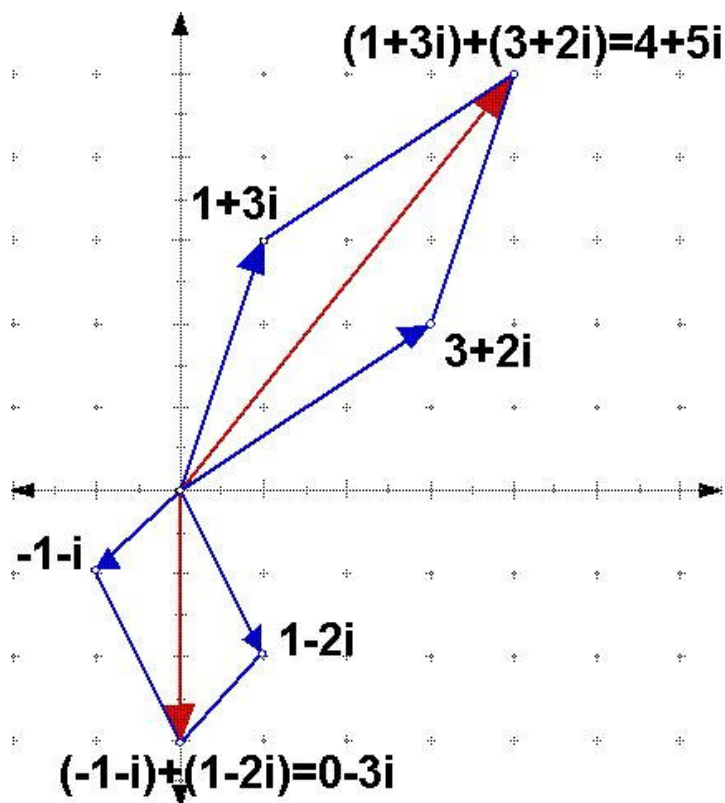
You add complex numbers by adding the real parts and then adding the imaginary parts.

Examples:

$$(3+2i) + (1+3i) = 4+5i$$

$$(-1-i) + (1-2i) = 0-3i$$

Here's how that looks in the complex plane:



SUBTRACTION

Subtraction of two complex numbers $a+bi$ & $c+di$ is defined as follows:

$$(a+bi) - (c+di) = (a-b) + (b-d)i$$

To subtract complex numbers, we subtract the real parts and the imaginary parts.

Examples:

$$(3+2i) - (1+3i) = (3-1)+i(2-3) = 2-i$$

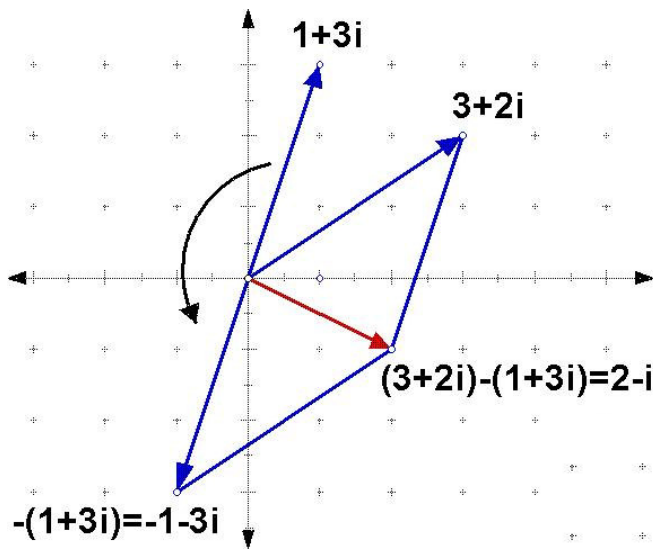
$$(-2-i) - (2+3i) = (-2-2)+i(-1-3) = -4-4i$$

When representing subtraction in the complex plane, it's the same as adding, except one thing.

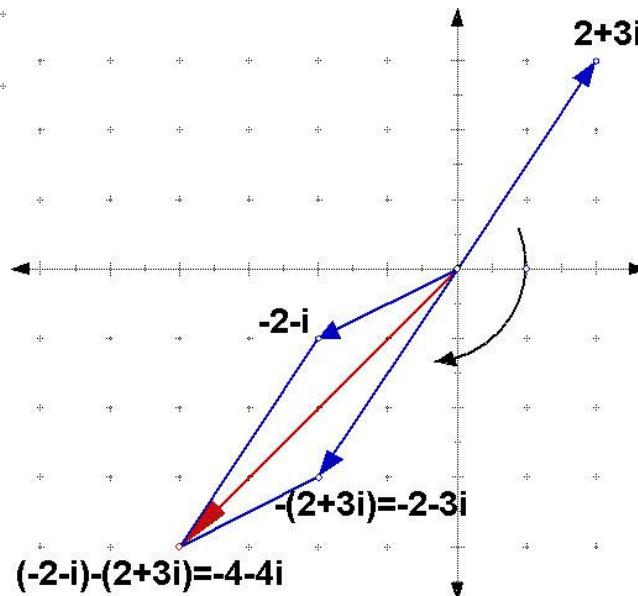
If you're subtracting, you have to turn the "arrow" of the second complex number around backwards, because simply if you're subtracting two complex numbers, you in fact add the first one to the negative form of the second one.

Here's how that looks in the complex plane:

$$(3+2i) - (1+3i) = (3-1)+i(2-3) = 2-i :$$



$$(-2-i) - (2+3i) = (-2-2)+i(-1-3) = -4-4i:$$



Absolute value = modulus of a complex number $z = a + bi$ can be defined as $|z| = \sqrt{a^2 + b^2}$.

Complex conjugate of a complex number $z = a + bi$ can be defined as $z' = a - bi$. It follows that the product zz' is therefore a wholly real number, since $zz' = (a + bi)(a - bi) = a^2 - iab + iab - i^2b^2 = a^2 + b^2$, since $i^2 = (\sqrt{-1})^2 = -1$.

De Moivre's theorem states how to find powers of a complex number. If we write a complex number in its trigonometric form $z = |z| (\cos(\Theta) + i \sin(\Theta))$, then $z^n = [|z| (\cos(\Theta) + i \sin(\Theta))]^n = |z|^n (\cos(n\Theta) + i \sin(n\Theta))$, when n is any positive integer.

De Moivre's theorem is a relatively simple formula for calculating powers of complex numbers.

$$z = r (\cos \varphi + i \sin \varphi); r = |z|; \operatorname{tg} \varphi = \frac{iy}{x} = \frac{b}{a}$$

Each complex number can be written in its exponential form as follows:

$$z = |z| (\cos \varphi + i \sin \varphi) = |z| e^{i\varphi}, \text{ where } e \text{ is Euler's constant.}$$

MULTIPLICATION

The multiplication of two complex numbers $a+bi$ & $c+di$ is defined as follows:

$$(a+bi)(c+di) = (a.c-b.d) + (a.d+bc)i$$

To multiply complex numbers we use the algebraic method which uses the "FOIL" method (multiply the first terms, then the outer terms, thirdly the inner terms, and the last terms).

Very important to remember is that $i^2 = -1$

Examples:

$$(1+i)*(2-i) = 1*2 + 1*(-i) + i*2 + i*(-i) = 2-i+2i-i^2 = 2-i+2i+1 = 3+i$$

$$(1+i)*(-1+i) = -1 + i - i + i^2 = -1 -1 = -2$$

$$i = \sqrt{-1}$$

$$i^2 = (\sqrt{-1})^2 = -1$$

$$i^3 = i^2 \cdot i = -i$$

$$i^4 = (\sqrt{-1})^4 = 1$$

$$i^5 = i$$

$$i^6 = -1$$

$$i^7 = -i$$

$$i^8 = 1$$

Example: $i^{35} = (i^4)^8 \cdot i^3 = -i$

DIVISION

Complex numbers are divided so we multiply the fraction with a 'unit in a special form'. This special unit is composed of a fraction, where both numerator and denominator are complex conjugate to the denominator

in the original fraction. E.g. $\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac-adi+bci+bd}{c^2+d^2}$

SUMMARY**COMPLEX NUMBERS**

- contain a real part and an imaginary part
- a set of all numbers of the standard form of complex numbers

A STANDARD FORM OF A COMPLEX NUMBER

- $z = a + bi$, where a is a real part and bi is the imaginary part

REAL PART

- contains real number a

IMAGINARY PART

- contains real number b and the imaginary unit i

IMAGINARY UNIT

- i defined as $i^2 = -1$ or as $i = \sqrt{-1}$

REAL NUMBERS

- when the imaginary part of a complex number is zero

PURELY IMAGINARY NUMBERS

- complex numbers whose real part is zero

COMPLEX PLANE

- a plane with 2 perpendicular axes on which a complex number is plotted at the coordinate system

REAL AXIS

- =x-axis, which represents the real part

IMAGINARY AXIS

- =y-axis, which represents the imaginary part

MODULUS

- = $\sqrt{a^2+b^2}$ for a complex number $a+bi$

- in the complex plane it is the distance between the complex number & the origin, in other words it is a measure of the magnitude of z , and it is written as $|z| = \sqrt{a^2+b^2}$

ARGUMENT

-it is the magnitude in radians of the angle between the positive part of x-axis & the arrow connecting the complex number with the origin

POLAR FORM

-of a complex number $z=a+bi$ is: $z = r (\cos(\theta) + i \sin(\theta))$, where $r=|z|$ and θ is the argument of z , i.e. the angle

POLE

=the fixed origin - the fixed point in the polar coordinate system from which every point is r units away