Factorial

A positive integer factorial is the product of each natural number up to and including the integer. If n is a positive integer, then

n! = n (n – 1) (n – 2) (n – 3) .... 4.3.2.1 (read as n factorial)

n! = n (n – 1)!

A special case is 0! = 1

Example: 4! = 4.3.2.1 = 24

3! = 3.2.1 = 6

$$\frac{8!}{4!} = \frac{8.7.6.5.4!}{4!} = 8.7.6.5 = 1680$$

Successive operations

If there are 3 paths joining A to B and 4 paths joining B to C, then there are 3.4 or 12 different ways of going from A, through B, to C.

If there are 2 paths joining C to D, then there are 3.4.2 or 24 ways of going from A, through B and C, to D.

If there are *r* ways of performing one operation, *s* ways of performing a second operations, *t* ways of performing a third operation and so on, then there are (*r.s.t.* ….) different ways of performing the operations in succession.

This multiplication rule only applies when the operations are independent, i.e. the choice made for one operation does not affect the choice made for any of the other operations.

Permutations => P = An arrangement of objects without repetition in a specific order

Example: If we have 3 letters A, B, and C, these can be written in a row in the following way: The first letter to be written down can be chosen in 3 ways. The second letter can then be chosen 2 ways and the remaining letter is written down in the third position. Thus the three operation can be performed in 3.2.1 or 6 ways.

This can be stated in general terms as follows:

The number of ways of arranging *n* different things in a row is n (n – 1) (n – 2) ….. 3.2.1. A useful shorthand way of writing this expression is *n!*

Def: A permutation of n objects, arranged into one group of size n, without repetition, and order being important is: P (n, n) = n! (i.e. *permutation using **all** the objects*)

Circular arrangements

If we want to arrange *n* people around a circular table, the number of possible arrangements will no longer be n! because there is no distinction between certain arrangements there were distinct when written in a row.

With circular arrangements, it is the relative positions of the items being arranged which is important. One item can therefore be fixed and the remaining items arranged around it. The number of arrangements of n unlike thinks in a circle will there fore be (n – 1)!

Distinguishable permutations = Permutations with a repetition of some items => P'

Sometimes items are repeated and all of the permutations aren't distinguishable from each other.

Example: Find all permutations of the letters BOB

There are really only three distinguishable permutations here: BOB, BBO, OBB.

Generally, if a word has N letters, k of which are unique and you let n ($n_1, n_2, \ldots, n_k$) be the frequency of each of the k letters, then the total number of distinguishable permutations is given by: $\frac{N!}{n1!n2!\ldots nk!} = P'_{n1, n2, \ldots, nk}(N)$

Ex.: Consider the word STATISTICS. The frequency of each letter: S = 3, T = 3, A = 1, I = 2, C = 1, there are 10 letters total, so the number of distinguishable permutations is: $\frac{10!}{3!.3!.1!.2!.1!} = 50\,400$

Permutations of objects selected from a group = Variations => V (k, n)

Suppose we want to arrange *k* objects chosen from *n* unlike objects: The first object can be chosen in n ways, the second object can be chosen in (n – 1) ways, the third object in (n – 2) ways and so on until the k^{th} item which can be chosen in (n – k + 1) ways. Thus the number of permutations of k objects chosen from n unlike objects is n (n – 1) (n – 2) (n – 3)…(n – k + 1)

$= \frac{n!}{(n-k)!} = V(k, n)$

So if we make a selection of objects from a group and we do not select all of the objects, we will use the term **variations**, instead of permutations (this term is exclusive for arrangement of all the objects from a group) in a specific order.

Variations with a repetition => V' (k, n) = n^k

Ex.: How many phone stations can be connected up, if the phone numbers consist of 5 digits? We have 10 figures, but the numbers cannot begin with 0.

n = 10, k = 5 V'(5, 10) = 10^5 = 100 000

Numbers, which begin with 0: V'(4, 10) = 10^4 = 10 000

So the number of phone stations is 100 000 – 10 000 = 90 000

Combinations => C

A combination is an arrangement of objects without repetition where order is **not** important.

Note: The only difference between a variation and a combination is whether order is important.

A combination of n objects, arranged in groups of size k, without repetition is: C (k, n) = $\frac{n!}{k!.(n-k)!}$

Another way to write a combination of n things, k at a time is using the *binomial notation*:

$\binom{n}{k}$ => Slovak equivalent could be a 'combination number'

Combinations with a repetition => C'(k, n) = $\binom{n+k-1}{k}$

There are 3 kinds of syrups in a shop: strawberry, raspberry and peach one. We shall buy 4 bottles of the syrup. How many possibilities do we have?

C'(4, 3) = $\binom{3+4-1}{4} = \binom{6}{4}$ = 15

The binomial expansion theorem

Combinations are used in the binomial expansion theorem from algebra to give the coefficients of the expansion $(a + b)^n$. They also form a pattern known as Pascal's Triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Each element in the table is the sum of the two elements directly above it. Each element is also a combination. The n value is the number of the row (start counting at zero) and the k value is the element in the row (start counting at zero).

$$\binom{n}{k}$$

$(a + x)^1 = a + x$

$(a + x)^2 = a^2 + 2ax + x^2$

$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$

$(a + x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$

Extracting the coefficients of *a* and *x* we obtain Pascal's Triangle.

Symmetry

Pascal's Triangle illustrates the symmetric nature of combination. C (k, n) = C (n – k, n)

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

$$\binom{0}{0} = 1$$

$$\binom{n}{0} = 1 \qquad \binom{n}{1} = n$$

Since combinations are symmetric, if n – k is smaller than k, then switch the combination to its alternative form and then use the shortcut given above.

The binomial expansion theorem – For any real numbers a, b and any $n \in N_0$ it is true that

$$(a + b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots\ldots + \binom{n}{k} a^{n-k} b^k + \ldots\ldots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

This is a binomial expansion of a binomial $(a + b)^n$, where the 'combination numbers' are coefficients of the binomials in this expansion.