ANALYTICAL GEOMETRY OF LINEAR SHAPES

Ex: Find out, whether the following points lie on the same line: A [3,3,0], B [5,4,3], C [7,5,6].

(AB, AC are linear dependent, so those three points lie on the same line.)

u = AB

u = B – A

A + **u** = B

If X is any point of the line set by points A, B, then it is possible to write:

A + t**u** = X, t ∈ R.

The equation X = A + t**u** is called the *vector (parametric) equation* of a line, when t ∈ R.

A = the fixed point of the line

u = directive vector of the line

Given X [x, y], A [a_1, a_2], **u** = (u_1, u_2) we may rewrite the equation in the form of coordinates:

$x = a_1 + tu_1$

$y = a_2 + tu_2$

Ex: Write the parametric equation of the line p, which passes through the point A [2,3] and its directive vector is **u** = (-1,5).

$x = a_1 + tu_1$

$\underline{y = a_2 + tu_2}$

x = 2 - t

y = 3 + 5t t∈R, t = parameter

If t = 1 then X [1,8]

t = -2 then Y [4,-7]

Check whether point Z [1,1] ∈ p. Then it should be true that 1 = 2 – t

$\underline{1 = 3 + 5t}$

Parameters t are not equal from these two equations, therefore Z doesn't lie on the line p.

If parameters are equal, then Z lies on p.

We may conclude that any point X is a sum of the fixed point of the line and t – multiple of the directive vector of that line.

Note: if $t \in R$, then we have the equation of a line

$t \geq 0$ then it is the equation of a half line which lies on the right side of A

$t \leq 0$ then it is the equation of a half line which lies on the left side of A

$m \leq t \leq n$ then it is the equation of an abscissa

Example:

$x = 1 + t$

$\underline{y = 2 + 3t}$

If t = ½ ….. S [3/2, 7/2]

Find a point C, which is an intersection of the line AB with the x- axis. > $c_2 = 0$

D, y- axis. > $d_1 = 0$

$x = 1 + t$

$\underline{0 = 2 + 3t}$ > $t = -2/3$

$x = 1/3$ C [1/3, 0]

$0 = 1 + t$ > $t = -1$

$\underline{y = 2 + 3t}$

$y = -1$ D [0,-1]

A vector which is perpendicular to a line is called the *normal vector* of the straight line.

Vector (parametric) equation of a straight line *in the space*:

$x = a_1 + tu_1$

$y = a_2 + tu_2$

$z = a_3 + tu_3, \quad t \in R$

Ex: Check whether the following equations are the parametric equations of the same straight line.

$x = 7 - 2t$ $x = 11 + 4s$

$y = 3t$ $y = -6 - 6s$

$$\left.\begin{array}{l} 11 = 7 - 2t \ldots t = -2 \\ -6 = 3t \ldots\ldots t = -2 \end{array}\right\}$$

the lines are identical/coincident, i.e. these equations are the expressions of the same line

By eliminating the parameter t from the parametric (vector) equations we will get the *Cartesian (general) equation* of a straight line in the plane **ax + by + c =0**, where at least one of the numbers a, b is different form 0.

E.g.

x = -1 – 2t …… t = $\frac{-1-x}{2}$

<u>y = 5 – 4t</u>

y = 5 – 4 $\frac{-1-x}{2}$

y = 5 – 2(-1 – x)

y = 7 + 2x

<u>0 = 2x – y + 7</u> Cartesian equation of the straight line, where the coefficients before x and y are the coordinates of the normal vector to this line, i.e. **n** (2, -1)

Each line in the plane has an infinite number of the Cartesian equations, which are non-zero multiples of each others.

THE CARTESIAN EQUATION OF A LINE IN A SPECIAL POSITION: ax + by + c = 0

	The position of a line by virtue of axes	**Conditions**	**Form of the Cartesian equation**
1	line passes through the origin of the axes	c = 0	ax + by = 0
2	line is parallel with the x- axis	a = 0	by + c = 0
3	line is parallel with the y- axis	b = 0	ax + c = 0
4	line is coincident with the x - axis	c = 0, a = 0	y = 0
5	line is coincident with the y-axis	c = 0, b = 0	x = 0

The other ways of the analytical expression of a line in the plane.

Given the Cartesian equation ax + by + c = 0, supposing that b $\neq 0$, we may write that

y = $\frac{-a}{b}x-\frac{c}{b}$

The equation **y = kx + q** of a line is called a *directive form of the line equation*; where **k** is the *directive = gradient.*

Directive of a line **k** is equal to the tangent of the direction angle of the line, i.e. the angle between the line and the positive part of the x-axis. (k = $\frac{-a}{b}$)

q is the segment on the y-axis, which the line intersects on it. (q = $-\frac{c}{b}$)

VECTOR EQUATION OF A PLANE

Through any three different points A, B, and C, which don't lie on one line, passes one plane only. The oriented abscissas **AB = u, AC = v** are linear independent.

For every ordered pair [t, s] ∈R x R is a point X = A + t**u** + s**v** a point from the plane ABC.

The equation X = A + t**u** + s**v,** where **AB = u, AC = v** is a *vector equation of a plane* α (A**, u, v).**

If X[x, y ,z], A[a_1, a_2, a_3], **u** (u_1, u_2, u_3), **v** (v_1, v_2, v_3), we may rewrite the vector equation of a plane through coordinates:

<u>X = A + t**u** + s**v**</u>

$x = a_1 + tu_1 + sv_1$

$y = a_2 + tu_2 + sv_2$

$z= a_3 + tu_3 + sv_3$ where t, s ∈R

Cartesian equation of a plane

Similarly as we expressed a line in a plane with its Cartesian equation, we may write the Cartesian equation of a plane in the space. Again, we start from its vector equations, and by eliminating parameters we get an equation in the form: *ax + by +cz +d = 0*, i.e. a linear equation with the variables x, y, z. Numbers a, b ,c are the coordinates of a vector **n = u** x **v** (i.e. we may get the coordinates of the normal vector as a vector product of the directive vectors u and v).

<u>Example</u>: x = 2 – t + 2s

y = 1 + 2t – s

<u>z = -3 + t + s</u>

x – 2 = -t + 2s /2

$\underline{y - 1 = 2t - s} \quad /2$

$2x + y - 5 = 3s$

$\underline{x + 2y - 4 = 3t}$

$z = -3 + t + s \quad /3$

$3z = -9 + 3t + 3s$

$\underline{3z = -9 + x + 2y - 4 + 2x + y - 5}$

$3z = -18 + 3x + 3y$

$z = -6 + x + y$

$\underline{0 = x + y - z - 6}$

Note: equations $ax + by + cz + d = 0$ where [a, b, c] $\neq$ [0, 0, 0] are expressions of planes in the space, not of lines. No line in the space has a Cartesian equation, i.e. although we eliminate a parameter from the vector equations of a line in the space, we will not get its Cartesian equation; this equation will be an expression of a plane which contains that line!!!

MUTUAL POSITIONS OF LINES AND PLANES

For any two lines p(A, **u**), q(B, **v**) in the space it is true that:

a) p || q ⇔ **v** = k . **u**
b) p = q ⇔ **v =** k **. u ^ AB =** k **. u**
c) if it is not true that p || q, then p, q intersect in the plane, or are skew lines in the space; in other words: p, q lie in the same plane ⇔ **AB** is a linear combination of **u, v**
p, q are skew lines ⇔ **AB** is not a linear combination of **u, v**, and **v** is not a multiple of **u**

Lines p, q are parallel if n_p is a multiple of n_q.

THE ANGLE OF TWO LINES

For the angle α of the lines p(A, **u**), q(B, **v**) it is true that:

$$\cos \alpha = \left| \frac{u.v}{|u|.|v|} \right|$$

Example: page 101- 103

MUTUAL POSITION OF TWO PLANES

Two planes α (A, u, u'), β (B, v, v') in the space may have one of the following three positions:

a) α, β are parallel and coincident ⇔ each of the vectors v, v', AB is a lin. combination of u, u'
b) α, β are parallel ⇔ both vectors v, v' are lin. combinations of u, u', but AB is not $[n_\alpha = k \cdot n_\beta]$
c) α, β intersect in one common line p ⇔ at least one of the vectors v, v' isn't a lin. combination of u, u' $[n_\alpha \neq k \cdot n_\beta]$

VECTOR EQUATION OF A LINE OF INTERSECTION

E.g. α: $2x - y + z + 1 = 0$ β: $x + y + 2z - 3 = 0$

We find two different points A, B of the line of intersection p, find the vector AB and then write the vector equation of that line p.

For instance, z = 0, then z = 2 and calculate x and y.

$z = 0$ …. $2x - y + 1 = 0$ …. $x = \frac{2}{3} \Rightarrow y = \frac{7}{3}$ …… $A[\frac{2}{3}, \frac{7}{3}, 0]$

$z = 2$…… $B[-\frac{4}{3}, \frac{1}{3}, 2]$

$AB[-2, -2, 2] = -2[1, 1, -1]$

p: $X = A + tu$

$x = \frac{2}{3} + t$

$y = \frac{7}{3} + t$

$z = 0 - t$ $t \in R$