POLYNOMIALS IN GENERAL

If $f(x) = \mathbf{ax^n + bx^{n-1} + cx^{n-2} + \ldots + k}$, $a \neq 0$, then f(x) is said to be a polynomial of degree (or order) **n** in the variable **x.**

If we consider the equation f(x) = 0 this will have up to **n** real roots. The number of real roots will depend upon the values of **a, b ,c,** and **d**.

Horner's scheme: the first line consists of the coefficients of the divisor; the second line contains the multiples of a number in the third line with the suspicious 'root'. The last line is the sum of a number in the 1st line with the number in the 2nd line.

$(x^4 – 2x^3 + 4x^2 – 6x + 8) : (x – 1) = x^3 – x^2 + 3x – 3 + 5/x – 1$

	1	-2	4	-6	8
1		1 . 1 = 1	-1 . 1 = -1	3 . 1 = 3	-3 . 1 = -3
	1	1 – 2 = -1	-1 + 4 = 3	3 – 6 = -3	-3 + 8 = **5**

The last window contains a remainder > 5 is the remainder; if there is zero in the last window, number 1 is a root. Numbers in the last line are the coefficients before x^3, x^2, x, x^0.

If the function f(x), a polynomial in x, is divided by (x – a) until the remainder does not contain x, i.e. it is a constant, then the remainder is equal to f(a) .

Proof :

Suppose that when **f(x)** is divided by **(x – a)**, the quotient is **g(x)** and the remainder is **k**, then

$f(x) \equiv (x – a) . g(x) + k$

Substituting x =a in both sides of this identity

f(a) = 0 . g(a) + k

f(a) = k

So the remainder when f(x) is divided by (x – a) us equal to f(a).

By a similar argument, when f(x) is divided by (cx – a) the remainder is f(a/c).

Example: Find the remainder when $8x^3 – 4x^2 + 6x + 7$ is divided by (x – 1)

Using the remainder theorem, when f(x) is divided by (x – 1), the remainder is f(1).

f(1) = 8 . 1 – 4 . 1 + 6 . 1 + 7 = 17

The remainder is 17.

RELATION OF DIVISION

a, b ∈ Z. We say that **a** divides **b**, if there exists such a number **c** ∈ Z, that **a . c = b**. If such a number **c** does not exist, we say that **a** does not divide **b**. Notation **a/b**.

E.g. –2/6 because there is a number –3, for which it is true that –2 . (-3) = 6.

If **a/b** then **a** is a divisor of **b** and **b** is a multiple of **a**.

It is true that:

1. a/0
2. 1/a
3. 0/a => a = 0
4. a/1 => a= 1, -1
5. a/a (reflectiveness)
6. a/b, b/c => a/c (transitiveness)

Division with a remainder: Let's have **a**, **b**∈ Z, and **b** $\neq 0$. Then there exists just one pair **q, r** ∈ Z only, for which it is true that **a = b . q + r,** and **0 ≤ r ≤ b**.

Number **q** is called a partial (fractional) *quotient*, **r** is called a *remainder*.

Let's have a number **b** which does not divide **a,** and at the same time **b** $\neq 0$. Then there exist such integers $\mathbf{q_1, r_2}$, for which it is true that $\mathbf{a = b \,.\, q_1 + r_2}$ and $\mathbf{0 \leq r_2 < b}$. We have to state that $0 < r_2$ – otherwise it would be true that b/a. Let's apply the formula for division with a remainder again; now for the pair $\mathbf{b, r_2}$ there exist such integers $\mathbf{q_2, r_3}$, for which

$\mathbf{b = r_2 \,.\, q_2 + r_3}$ and $\mathbf{0 < r_3 < r_2}$. If $\mathbf{r_3 = 0}$ we finish. If $\mathbf{r_3 \neq 0}$ we may apply the formula again for the pair $\mathbf{r_2, r_3}$. The procedure will be repeated with next pairs of remainders. Since the remainders **$\mathbf{r_2}$, r3, …** are descending non-negative integers, we are sure that there will occur a zero remainder. This formula is called Euclides's algorithm of long division:

$\mathbf{a = b \,.\, q_1 + r_2}$ $\mathbf{0 < r_2 < b}$

$\mathbf{b = r_2 \,.\, q_2 + r_3}$ $\mathbf{0 < r_3 < r_2}$

$\mathbf{r_2 = r_3 \,.\, q_3 + r_4}$ $\mathbf{0 < r_4 < r_3}$

.

$\mathbf{r_{n-1} = r_n \,.\, q_n}$

The last non-zero remainder in the Euclides' algorithm of division for the pair of numbers **a, b** is *the highest common factor* of **a** and **b**.

Example: Find the hcf of 2100, and 2002.

2100 = 2002 . 1 + 98

2002 = 98 . 20 + 42

98 = 42 . 2 + 14

42 = 14 . 3

So the hcf (2100, 2002) = 14

Ex: Let's have two polynomials:

$P(x) = x^4 + x^3 - 3x^2 - 4x - 1$

$Q(x) = x^3 + x^2 - x - 1$

$(x^4 + x^3 - 3x^2 - 4x - 1) : (x^3 + x^2 - x - 1) = x$, $k = -2x^2 - 3x - 1$

$(x^3 + x^2 - x - 1) : (-2x^2 - 3x - 1) = -x/2 + ¼$, $k = -3/4 . (x+1)$

$(-2x^2 - 3x - 1) : (x + 1) = -2x - 1$, $k = 0$

The last non-zero remainder is $x + 1$ = hcf