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Indices and Surds

The term **indices** refers to the **power** to which a number is raised. Thus x^2 is a number with an index of 2. People prefer the phrase "**x to the power of 2**".

Term **surds** is not often used, instead term **roots** is used. Occasionally you will be asked to give an answer in **surd form**; this implies that you should give the answer in terms of constants and square roots instead of working out an imprecise decimal approximation.

We know that 64 = 2 x 2 x 2 x 2 x 2 x 2 = 2^6

Here 2 is called **the base** and 6 is called **the power** (or **index** or **exponent**).

We say that "**64 is equal to base 2 raised to the power 6**".

x^3 is a shortening of $x \times x \times x$.

In the same way, any number to the power of *n* is that number multiplied by itself *n* times. To describe this in more detail, in the expression x^3, the *x* is referred to as the **base**, and the 3 as the **exponent.**

Similarly, if *m* is a positive integer and a $\neq$ 0 then a $\times$a $\times$a $\times$…*m times* = a^m

If m is a positive integer, a x a x a …. m times is written as a^m.

a is called **the base** and m is **the power**. We read it as "**a raised to the power m**".

The power is also called "the index" or "the exponent".

When the power or index is a fraction say $\frac{1}{n}$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$ and it is **a surd of order n**.

But this will be dealt with later on.

The Laws on Indices

The rules that are going to be suggested below are known as the laws on indices and can be written as:

1. Multiplication: $x^a x^b = x^{a+b}$

When you multiply indices you ***add*** the exponents together.

$$x^3 \times x^2 = (x \times x \times x) \times (x \times x) = (x \times x \times x \times x \times x) = x^5$$

As you can see $x^3 \times x^2 = x^{3+2} = x^5$.

2. Division: $\frac{x^a}{x^b} = x^{a-b}$

Division is, as expected, the opposite to multiplication. When you divide indices you ***subtract*** the exponents from each other.

$$\frac{x^4}{x^2} = \frac{x \times x \times x \times x}{x \times x} = \frac{\not{x} \times \not{x} \times x \times x}{\not{x} \times \not{x}} = x \times x = x^2$$

Again this shows that $\frac{x^4}{x^2} = x^{4-2} = x^2$

3. Negative powers: $x^{-n} = \frac{1}{x^n}$

Using the trusted method of showing each x separately we obtain:

$$\frac{x^2}{x^3} = \frac{\not{x} \times \not{x}}{\not{x} \times \not{x} \times x} = \frac{1}{x}$$

Using the just demonstrated method of subtracting the exponents:

$$\frac{x^2}{x^3} = x^{2-3} = x^{-1}$$

This implies that ***x to the power of a negative number*** is ***one divided by x to the power of that positive number.*** The specific case of $\frac{1}{x} = x^{-1}$ is referred to as the **reciprocal** of x or more often as the **inverse** of x.

4. Base raised to two powers: $(x^a)^b = x^{ab}$

When you have a base raised to two powers you ***multiply*** the powers.

$$\left(x^2\right)^3 = (x \times x) \times (x \times x) \times (x \times x) = x \times x \times x \times x \times x \times x = x^6.$$

As you can see that $\left(x^2\right)^3 = x^{2\times 3} = x^6$.

5. Multiple bases to the same power:

$$(xy)^n = x^n y^n$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

When you have two bases to the same power, you can ***raise both bases to the same power*** and multiply them.

For example:

$(xy)^3 = xy \times xy \times xy = x \times y \times x \times y \times x \times y = x \times x \times x \times y \times y \times y$ which is the same as x^3y^3.

Here is an example with numbers: $(2 \times 5)^2 = (10)^2 = 100 = 4 \times 25 = 2^2 \times 5^2$.

There is a similar situation with division: $\left(\frac{x}{y}\right)^2 = \frac{x}{y} \times \frac{x}{y} = \frac{x \times x}{y \times y} = \frac{x^2}{y^2}$

6. Fractional powers:

$$x^{\frac{a}{b}} = \sqrt[b]{x^a}$$

What if the power isn't even an integer? Suppose you wanted to find $x^{\frac{1}{2}}$, you could say that $x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^1$ (by law 1, addition of powers) which means that $x^{\frac{1}{2}}$ must be $\pm\sqrt{x}$. However it is customary to only use the positive root and so $x^{\frac{1}{2}}$ is defined as $\sqrt{x}$. You can use a similar argument for other such fractions, for example $\left(x^{\frac{1}{3}}\right)^3 = x$ so $x^{\frac{1}{3}} = \sqrt[3]{x}$. In cases when the numerator is not 1 we need to use other laws of indices to prove the square definition, for example $x^{\frac{2}{3}} = \left(x^2\right)^{\frac{1}{3}}$ (using law 4), and $\left(x^2\right)^{\frac{1}{3}} = \sqrt[3]{x^2}$ (using the definition above). It's useful to remember the general rule that $x^{\frac{a}{b}} = \sqrt[b]{x^a}$.

7. The 0 and 1 Power: $x^0 = 1$

$$x^1 = x$$

You may well have realised that $x^1 = x$. You can prove this by $\frac{x^n}{x^{n-1}}$ which is clearly x which is $x^{n-(n-1)} = x^1 = x$ by law 2.

Also with x^0 we can prove that it is equivalent to 1, $\frac{x^n}{x^n} = 1$, but by law 2 it is also equivalent to $x^{n-n} = x^0$

Basic Rule of Radicals: Roots/Surds

Expressions such as $\sqrt{4}$, $\sqrt{25}$ … have exact numerical values, viz. $\sqrt{4}$ = 2, $\sqrt{25}$ = 5. These expressions $\sqrt{4}$, $\sqrt{25}$ are called **roots**.

But expressions such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ … cannot be written as numerically exact quantities.

For example we might say that $\sqrt{2}$ = 1,4 correct to 2 s.f.

Or that $\sqrt{2}$ = 1,4142136 correct to 8 s.f.

But we can never find an exact quantity equal to $\sqrt{2}$.

Such numbers are called **irrational** and it is often convenient to leave them in the form $\sqrt{2}$ and they are then called **surds**.

Since $\sqrt[n]{a}$ is an nth root, it is called a surd of order n, if it is *irrational*. E.g.

$\sqrt[3]{10}$ is a surd of order 3

$\sqrt{53}$ is a surd of order 2

$\sqrt[4]{81}$ is NOT a surd because $\sqrt[4]{81} = 3$, and 3 is NOT an irrational number.

As surds frequently occur in solutions it is useful to be able to simplify them. To simplify means to find another expression with the same value. It does not mean to find a decimal approximation.

Because $\sqrt{x} \times \sqrt{x} = x$, it is useful to know that it can be rearranged to give $\sqrt{x} = \frac{x}{\sqrt{x}}$ and $\frac{1}{\sqrt{x}} = \frac{\sqrt{x}}{x}$.

Surds as indices

Because $\sqrt[n]{x} = x^{\frac{1}{n}}$ the laws of indices also apply to any n-th root. The most frequently used instances of this is law 5 with surds:

- $(xy)^{\frac{1}{2}} = x^{\frac{1}{2}} y^{\frac{1}{2}}$ becomes $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$
- $\left(\frac{x}{y}\right)^{\frac{1}{2}} = \frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}$ becomes $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

The first of these points is often used to simplify a surd, for example $\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} = 10\sqrt{2}$. In an exam, you will be expected to write all surds with the smallest possible number inside the surd (i.e. the number inside the root shouldn't have any square factors).

Note: This rule only applies to non-negative numbers, attempts to involve negative numbers might give absurd results; $-1 = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$

How to simplify (reduce) a radical?

1. Find the **largest** perfect square which will divide evenly into the number under your radical sign. This means that when you divide, you get no remainders, no decimals, and no fractions.

Reduce: $\sqrt{48}$ the largest perfect square that divides evenly into 48 is 16.

(If the number under your radical cannot be divided evenly by any of the perfect squares, your radical is already in simplest form and cannot be reduced further.)

2. Write the number appearing under your radical as the product (multiplication) of the perfect square and your answer from dividing.

$$\sqrt{48} = \sqrt{16 \cdot 3}$$

3. Give each number in the product its own radical sign.

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3}$$

4. Reduce the "perfect" radical which you have now created.

$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}$

5. And the answer is $\sqrt{48} = 4\sqrt{3}$.

<u>What happens if you do not choose the largest perfect square to start the process?</u>

If instead of choosing 16 as the largest perfect square to start this process, you choose 4, look what happens.....

$\sqrt{48} = \sqrt{4 \cdot 12}$

$\sqrt{48} = \sqrt{4 \cdot 12} = \sqrt{4} \cdot \sqrt{12} = 2\sqrt{12}$. Unfortunately, this answer is **not** in simplest form.

The 12 can also be divided by a perfect square (4).

$2\sqrt{12} = 2\sqrt{4 \cdot 3} = 2\sqrt{4} \cdot \sqrt{3} = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$

If you do not choose the largest perfect square to start the process, you will have to repeat the process; still if you do not make a mistake, you will get the correct solution.

The example shown above on radicals shows **ALL** of the steps in the process. It may **NOT** be necessary for you to list **EVERY** step. As long as you understand the process and can arrive at the correct answer, you are **correct!**

Addition and Subtraction of Radicals

When adding or subtracting radicals, you must use the same concept as that of adding or subtracting like variables. In other words, the radicals must be the same before you add (or subtract) them.

Example 1:

$2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}$

Since the radicals are the same, simply add the numbers in front of the radicals (do NOT add the numbers under the radicals).

Example 2:

$4\sqrt{5} + 3\sqrt{7}$

Since the radicals are not the same, and both are in their simplest form, there is no way to combine them. The answer is the same as the problem.

Warning: If the radicals in your problem are different, be sure to check to see if the radicals can be simplified. Often times, when the radicals are simplified, they become the same radical and can then be added or subtracted. Always simplify, if possible, before deciding upon your answer.

Example 3:

$5\sqrt{3} + 2\sqrt{75} = 15\sqrt{3}$

How did I calculate it? Look at the following steps:

1. Are the radicals the same?	**Answer:**	**NO**
2. Can we simplify either radical?	**Answer:**	**Yes,** $2\sqrt{75}$ **can be simplified.**
3. Simplify the radical.	**Answer:**	$2\sqrt{75} = 2\sqrt{25 \cdot 3} = 2\sqrt{25}\sqrt{3}$ $2 \cdot 5 \cdot \sqrt{3} = 10\sqrt{3}$
4. Now the radicals are the same and we can add.	**Answer:**	$5\sqrt{3} + 10\sqrt{3} = 15\sqrt{3}$

Example 4:

Simplify: $5\sqrt{8} - 3\sqrt{18} + \sqrt{3}$

1. Simplify $5\sqrt{8}$ **Answer:** $5\sqrt{8}=5\sqrt{4\cdot 2}=5\sqrt{4}\sqrt{2}=5\cdot 2\sqrt{2}=10\sqrt{2}$

2. Simplify $-3\sqrt{18}$ **Answer:** $-3\sqrt{18}=-3\sqrt{9\cdot 2}=-3\sqrt{9}\sqrt{2}=-3\cdot 3\sqrt{2}=-9\sqrt{2}$

3. Since the radicals in steps 1 and 2 are now the same, we can combine them. $10\sqrt{2}-9\sqrt{2}=\sqrt{2}$

4. You are left with: $\sqrt{2}+\sqrt{3}$

5. Can you combine these radicals? **Answer: NO**

6. Therefore, Answer: ------------------> $\sqrt{2}+\sqrt{3}$

Multiplication and Division of Radicals

When multiplying radicals, one must multiply the numbers OUTSIDE (O) the radicals AND then multiply the numbers INSIDE (I) the radicals.

$$O\sqrt{I}\cdot O\sqrt{I}=O\cdot O\sqrt{I}\cdot\sqrt{I}$$

When dividing radicals, one must divide the numbers OUTSIDE (O) the radicals AND then divide the numbers INSIDE (I) the radicals.

$$\frac{O\sqrt{I}}{O\sqrt{I}}=\frac{O}{O}\cdot\sqrt{\frac{I}{I}}$$

Example 1:

Multiply and simplify: $2\sqrt{18}\cdot 3\sqrt{8}=72$

1. Multiply the outside numbers first $2\cdot 3=6$
2. Multiply the inside numbers $\sqrt{18}\cdot\sqrt{8}=\sqrt{144}$
3. Put steps 1 and 2 together and simplify $6\sqrt{144}=6\cdot 12=72$

Example 2:

Divide and simplify: $\frac{-12\sqrt{24}}{3\sqrt{2}}$

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1. Divide the outside numbers first. $\frac{-12}{3} = -4$

2. Divide the inside numbers. $\frac{\sqrt{24}}{\sqrt{2}} = \sqrt{12}$

3. Put steps 1 and 2 together and simplify. $-4\sqrt{12} = -4\sqrt{4}\sqrt{3} = -4\cdot 2\sqrt{3} = -8\sqrt{3}$

4. Therefore, the answer is: $-8\sqrt{3}$

Rationalising the denominator

A fraction that contains a radical in its denominator can be written as an equivalent fraction with a *rational denominator*.

Never leave a radical in the denominator of a fraction. Always ***rationalise the denominator***. This means removing a surd from a fraction. We **rationalise** an expression so that we multiply it with number 1 in special form.

1. When the denominator is a monomial *(one term)*, multiply both the numerator and the denominator by the number occurring in the denominator; the fraction can then be simplified so that it no longer contains a radical.

Example 1: Simplify $\frac{2}{\sqrt{7}}$

$$\frac{2}{\sqrt{7}} \bullet \frac{\sqrt{7}}{\sqrt{7}} =$$

Multiplying the top and bottom by $\sqrt{7}$ will create the smallest perfect square under the square root in the denominator.

$$\frac{2\sqrt{7}}{\sqrt{49}} =$$

$$\frac{2\sqrt{7}}{7} \quad \textit{Answer}$$

Replacing $\sqrt{49}$ by 7 rationalises the denominator.

2. When there is more than one term in the denominator, the process is a little tricky. You will need to multiply the denominator by its ***conjugate***. The *conjugate* is the same expression as the denominator but with the opposite sign in the middle. In this case in the denominator we apply the standard difference of two squares expansion: (a – b) . (a + b) = $a^2 - b^2$

Example 2: Simplify $\frac{2}{5+\sqrt{3}} = \frac{2}{5+\sqrt{3}} \cdot \frac{5-\sqrt{3}}{5-\sqrt{3}} = \frac{2.(5-\sqrt{3})}{(5+\sqrt{3}).(5-\sqrt{3})} = \frac{2.(5-\sqrt{3})}{5^2-\sqrt{3}^2} =$

$\frac{2.(5-\sqrt{3})}{25-3} = \frac{2.(5-\sqrt{3})}{22} = \frac{5-\sqrt{3}}{11}$

Example 3:

$$\frac{2}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{2\left(\sqrt{3}+1\right)}{3-1} = \sqrt{3}+1$$

Common mistakes

Splitting up roots

A common mistake is to split $\sqrt{x+y}$ into $\sqrt{x}+\sqrt{y}$ or $(x+y)^2$ into $x^2 + y^2$, usually whilst moving it to the other side of the equals. Trying a few examples will quickly convince you that this is not possible:

- $\sqrt{64} \neq \sqrt{32} + \sqrt{32}$
- $\sqrt{25} \neq \sqrt{12} + \sqrt{13}$

Exercises:

a) $36^{-1/2}$

$= (6^2)^{-1/2} = 6^{-1}$

$= \frac{1}{6}$

b) $(8x^2)^{1/3} \div x^{-1/3}$

$= \frac{(8)^{1/3} \cdot x^{2/3}}{x^{-1/3}}$

$= 2.x^{\frac{2}{3}+\frac{1}{3}} = 2x$

c) **Evaluate:** $\sqrt[4]{\frac{256}{625}}$

Given expression $\left(\frac{256}{625}\right)^{1/4} = \left(\frac{4^4}{5^4}\right)^{1/4}$

$= \frac{4^{4\times 1/4}}{5^{4\times 1/4}}$

$= \frac{4}{5}$

Practise:

(1) $\frac{\sqrt[3]{a^2b}}{\sqrt[3]{a^{-1}b^4}}$

(2) If $2^{x+2} = 128$, find the value of x.

(3) Simplify: $4x^{-3}y^2 \div (8xy)^2$

(4) Simplify: $\sqrt[4]{x^{3a}\ y^6} \times (x^{2/3} \times y^{-1})^a$

Solutions

(1) Given expression: $\frac{(a^2b)^{1/3}}{(a^{-1}b^4)^{1/3}} = \frac{a^{2/3} . b^{1/3}}{a^{-1/3} . b^{4/3}}$

$= \frac{a^{2/3+1/3}}{b^{4/3-1/3}} = \frac{a}{b}$

(2) Since $128 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

$= 2^7$

We have $2^{x+2} = 2^7$ [∵ bases are equal]

x + 2 = 7 [∴ powers are equal]

x = 5

(3) $\dfrac{4x^{-3}y^2}{(8xy)^2} = \dfrac{4}{x^3} \cdot \dfrac{y^2}{64x^2y^2}$

$= \dfrac{y^2}{x^3 . 16x^2y^2}$

$= \dfrac{1}{16x^5}$ $[\because y^2 \div y^2 = y^{2-2} = y^0 = 1]$

(4) $= \sqrt[4]{x^{3a}\ y^6} \times (x^{2/3} \times y^{-1})^a$

$= (x^{3a}\ y^6)^{1/4} \times (x^{2/3}\ y^{-1})^a$ $[\because \sqrt[4]{a} = a^{1/4}]$

$= x^{3a/4}\ y^{3/2} \times x^{2a/3}\ y^{-a}$ [∵ $(x^{3a})^{1/4} = x^{3a/4}$ (power law)]

$= x^{\frac{3a}{4}+\frac{2a}{3}} \times y^{\frac{3}{2}-a}$

$= x^{\frac{17a}{12}} \times y^{\frac{3}{2}-a}$

Summary

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^m = a^m\ b^m$
- $a^{m/n} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$
- $a^0 = 1$
- $a^{-1} = \dfrac{1}{a}$
- $a^{-n} = \dfrac{1}{a^n}$